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NÍCOLAS PIMENTEL DE SOUZA

POTENTIAL DYNAMIC GAMES WITH TIMING FRICTIONS

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Dissertação apresentada à Escola de Economia de São Paulo como pré-requisito à obtenção de título de mestre em Economia de Empresas.

Orientador: Bernardo de Vasconcellos Guimarães.

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Resumo

Nesse presente trabalho, nós propomos duas alternativas para tornar o modelo do Frankel and Pauzner [2000] analiticamente mais tratável. Primeiro, nós mostramos que é possível resolver recursivamente o modelo ao maximizar a função de utilidade de um único agente representativo, o que nos permite, portanto, aplicar os ferramentais de programação dinâmica. Além disso, é proposto um método numérico que nos ajudaria a realizar algumas estáticas comparativas quando os choques presentes no modelo são relativamente pequenos em relação às fricções.

Palavras-chave: Função potencial, jogos dinâmicos e estocásticos.

Abstract

In this dissertation, we propose two distinct approaches in order to improve the Frankel and Pauzner [2000] model tractability. We first show that, using the notion of potential functions, we could solve the model recursively by maximizing the utility of a single representative agent, allowing us, therefore, to use the dynamic programming toolbox. Additionally, we propose a numerical method that would permit us do some comparative statics when the shocks are relatively small compared to the frictions.

Keywords: Potential functions, stochastic and dynamic games.

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1 Introduction

Coordination problems emerge in a variety of situations, from the simple action of crossing a street or choosing which social network to use, to more sophisticated ones such as firms' investment decisions and bank runs. In other words, we are constantly facing situations in which not only the state of nature matters, but also how our peers or competitors behave.

A framework for dealing with these problems was developed by Frankel and Pauzner [2000], F-P henceforth. They model an economy with a continuum of agents who choose between two possible actions and receive opportunities to revise their choices according to a Poisson clock. The state of nature or the economy's fundamental are continually changing and the actions of any economic agent has impacts on others. They show that in the presence of shocks, there is a unique equilibrium, i.e., agents choose their actions based on a single threshold that depends on this economic fundamental and on how many other players are taking the same action.

As mentioned before, this framework allows for numerous applications. For instance, Frankel and Burdzy [2005] and Guimaraes and Machado [2017] analyze business cycles in light of this type of model. They show that recessions may occur even in the presence of small shocks. This happens because, due to strategic complementarities, firms might not have enough incentives to invest if other firms are also not investing. Analogously, booms can occur even in a not so favorable economic scenario. That is, it might be reasonable for an individual firm to invest if it observes other firms doing so, because it may be indicative of future expansion.

Further, this framework can also be useful for investigating currency crisis events. Guimaraes [2006] and Daniëls [2009], for example, model investors' decisions on whether to attack or not a pegged currency. In line with the business cycle models, agents are concerned not only with the economy's fundamental, here represented by the country's stock of foreign reserves for example, but also with the behavior of other investors. There is also some related work done in liquidity provision and in network externalities, as seen in He and Xiong [2012] and in Guimaraes and Pereira [2016], respectively.

However, as Angeletos and Liao mention in the Handbook of Maacroeconomics, "[It is] somewhat surprising that this approach has not attracted more attention in applied research". Given the importance of the tools that are present in this framework (Dynamic environment with fundamental shocks and Calvo-like frictions) it is surprising that there is little work building on F-P, especially in macroeconomics.

We think that part of the reason for this relative lack of attention is the model tractability. As it is modeled, dealing with an individual agent's utility function is far from trivial due to its continually changing arguments.

The model's limited tractability has, up to now, restricting the scope of applications. So far, the theoretical results have been limited to some comparative statics and the contrast between the general equilibrium and the benevolent central planner equilibrium, as seen in Guimares et al. [2017], both when shocks are minimal.

In other words, applications of the F-P game have been made when one of the most important features of the model was absent, i.e, its dynamics. Thus, in order to deal with it, we explore a different plan of attack than the ones existing in the literature, aiming to increase the model tractability.

This dissertation continues as follows. Section 2 presents the framework. In section 3 we show that the agent's problem is equivalent to a modified planner's problem. Section 4 presents the methodology for solving the problem recursively. Section 5 discusses the result on bifurcation probabilities, and section 6 concludes.

2 Preliminaries

2.1 Framework

Time is continuous and runs forever. There is a continuum of agents with unit mass indexed by i. There are two possible actions $a_i \in \{0,1\}$. The instantaneous payoff of action 0 is given by a function $u_0(\theta_t, n_t)$ and the instantaneous payoff of action 1 is characterized as $u_1(\theta_t, n_t)$, both continuously differentiable. We define the instantaneous relative gain of action 1 as

$$\Delta u(\theta_t, n_t) = u_1(\theta_t, n_t) - u_0(\theta_t, n_t),$$

which is increasing in both arguments, and n_t is the measure of agents locked in action 1 at time t.

Agents receive opportunities to revise their actions according to a Poisson clock (independent across agents) with arrival rate δ. Once they choose an action they are locked in the chosen action until the Poisson shock hits again. θ follows a stochastic process, with drift μ and variance σ^2.

$$d\theta_t = \mu dt + \sigma dZ_t,$$

where Z_t is the standard Brownian motion.

The discounted expected gain of choosing 1 instead of zero at some date τ is given by

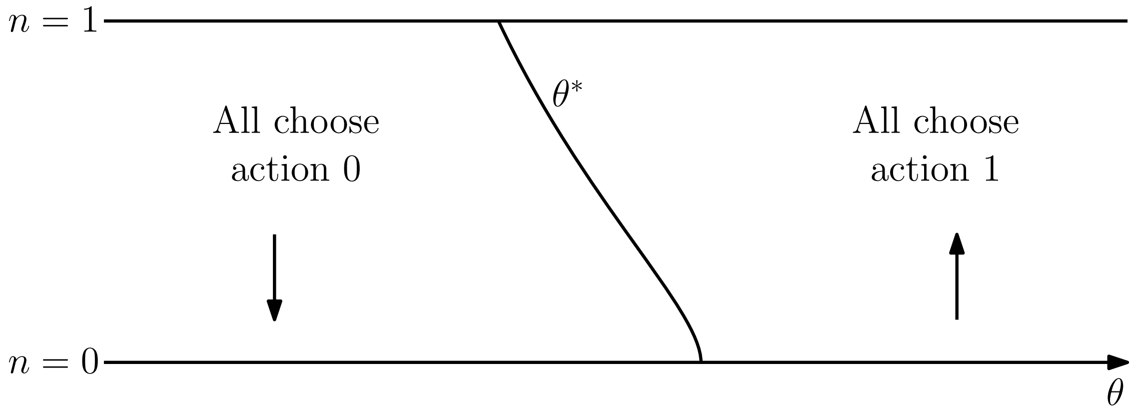
$$V_\tau = \int_\tau^\infty e^{-(\rho+\delta)(t-\tau)} \mathbb{E}\left[\Delta u(\theta_t, n_t)\right] dt, \tag{2.1}$$

where $\rho > 0$ is the discount rate of agents.

Frankel and Pauzner [2000] main result shows that there is a unique equilibrium in the presence of shocks, even when $\sigma \to 0$, characterized by a decreasing threshold $\theta^*(n)$. Notice that, without the presence of shocks, there is a multiple equilibrium region where the equilibrium depends on the expectations that agents form with respect to others' decisions. Intuitively, when shocks are present, the multiple equilibrium region shrinks to a single threshold since, when the state of the economy is close to the pessimistic or the optimistic dominance regions, agents cannot be indifferent anymore as they would probably fall into a dominated region in a short period of time.

Despite it was shown that this dynamic game can be fully characterized by a single decreasing threshold $\theta^*(n)$, no one has ever demonstrated - besides the limiting cases when there is vanishing frictions - how this threshold is related to the instantaneous utility

Figure 1 – Equilibrium threshold



function and the other parameters, such as the fundamental's variance (σ^2), the decision revision arrival rate (δ), and the discount rate (ρ).

Notice that this step might be crucial for future applications, and solving it from the perspective of a single agent may be impracticable, since, when facing a revision opportunity, she must take into account not only the current state of the economy and the expected time she will be locked in her decision, but also how the other infinitely many agents will behave before she can revise her action again.

One of the complications of this problem is that, because of the frictions, the 1st and 2nd welfare theorems fails, implying that there is no equivalence between the game solution and the solution derived from the maximization of the benevolent central planner utility.

Then, in order to overcome this difficulty we show that if one modify the central planner's utility by eliminating part of the externalities present in the game, we can solve this modified planner problem instead of the agent's problem, without loss of generality.

2.2 Potential Games

To do this, we use the notion of Potential Games, as seen in Monderer and Shapley [1996] for the finite population case, and in Sandholm [2001] for the continuous population case.

2.2.1 Finite Population Case

Definition 2.1. *A game $G = (N, \boldsymbol{A} = A_1 \times \cdots \times A_N, u : \boldsymbol{A} \to \mathbb{R}^N)$ is an* ***ordinal potential game*** *if there is a function $\boldsymbol{P} : \boldsymbol{A} \to \mathbb{R}$ such that $\forall a_{-i} \in A_{-i}$, and $\forall a_i^{'}, a_i^{''} \in A_i$,*

$$u_i(a^{'}, a_{-i}) - u_i(a^{''}, a_{-i}) > 0 \Longleftrightarrow P(a^{'}, a_{-i}) - P(a^{''}, a_{-i}) > 0$$

Example 2.1. Consider the Prisoner's dilemma described below, and let $P : \mathbf{A} \to \mathbb{R}$ be defined as

$$P(c,c) = 1$$
$$P(c,d) = 2$$
$$P(d,c) = 2$$
$$P(d,d) = 6$$

Figure 2 – Prisoner's Dilemma

Payoffs	Cooperate	Defect
Cooperate	(-1,-1)	(-10,0)
Defect	(0,-10)	(-6,-6)

Potential	Cooperate	Defect
Cooperate	1	2
Defect	2	6

Notice that

$$u_i(a', a_{-i}) - u_i(a'', a_{-i}) = P(a', a_{-i}) - P(a'', a_{-i})$$

Moreover, maximizing the potential function leads us to the pure strategy Nash equilibrium, that is, the potential function works, at least, as a refinement tool to the equilibrium set.

2.2.2 Continuous Population Case

Definition 2.2. *Let the set of populations be denoted by $P = \{1, \cdots, r\}$, where population p has mass m^p. The set of strategies for population p is denoted by $S^p = \{1, \cdots, n^p\}$. Further, the set of strategy distributions within population p is represented by $X^p = \{x \in \boldsymbol{R}_+^{n^p} : \sum_i x_i^p = m^p\}$, wilhe $X = \{x = (x^1, \cdots, x^r) \in \boldsymbol{R}_+^n : x^p \in X^p\}$ is the set of overall strategy distributions.*

Let $\bar{X} = \{x \in \boldsymbol{R}_+^n : m^p - \varepsilon \leqslant \sum_i x_i^p \leqslant m^p + \varepsilon, \forall p \in P\}$, where ε is a positive constant, and let the payoff function for strategy $i \in S^p$ be denoted by $F_i^p : \bar{X} \to \boldsymbol{R}$, such that $F : \bar{X} \to \boldsymbol{R}^n$ denote the vector of all payoff functions.

*Therefore, we say F is a **potential** game if there exists a C^1 function $f : \bar{X} \to \boldsymbol{R}$ such that $\frac{\partial f}{\partial x_i^p}(x) = F_i^p$, for all $x \in \bar{X}$, $i \in S^p$, and $p \in P$.*

3 Potential Dynamic Games

3.1 Frankel and Pauzner Potential Function

Since our game has an unique equilibrium, if we find its potential function, then we could solve the *F-P* game by maximizing it. Which is a much simpler task than trying to solve from the perspective of a single atomistic agent. That is what we show in the following proposition.

Proposition 3.1. *Let* $P(\theta_t, n_t) = \int_0^{n_t} \Delta u(\theta_t, v)dv$. *Then,*

$$\mathcal{P} = \max_{\phi_t \in [0,1]} \quad \mathbb{E}\left[\int_{\tau}^{\infty} e^{-\rho(t-\tau)} P(\theta_t, n_t)dt\right]$$

subject to

$$d\theta_t = \mu dt + \sigma dZ_t$$
$$dn_t = \delta(\phi_t - n_t)dt$$

solves the F-P game

Proof. Recall that agents receive opportunities to revise their actions according to a Poisson clock with arrival rate δ. Then, in order to maximize the potential function

$$\mathbb{E}\left[\int_{\tau}^{\infty} e^{-\rho(t-\tau)} P(\theta_t, n_t)dt\right] \tag{3.1}$$

we choose, at each time t, the proportion $\phi_t \in [0, 1]$ of agents who received a chance to switch actions and will pick action 1.

Suppose that at a given time τ it is optimal to choose $\phi_\tau < 1$ and consider the following deviation: we increase ϕ_τ by $d\phi > 0$ units today, but we keep the future values of ϕ_t unchanged, for any realization of the Brownian path. Increasing ϕ_τ by $d\phi$ today, raises n_τ by $\delta d\phi$. Notice, however that at a given time $t > \tau$ a proportion $e^{-\delta(t-\tau)}$ has already been selected again. Therefore, the change at time $t \geq \tau$ is:

$$dn_t = \delta d\phi e^{-\delta(t-\tau)}$$

This deviation is not profitable if:

$$\mathbb{E}\left[\int_{t=\tau}^{\infty} e^{-\rho(t-\tau)} \frac{dP(\theta_t, n_t)}{d\phi} dt\right] \leq 0$$

Notice that
$$\frac{dP(\theta_t, n_t)}{d\phi} = \frac{\partial P(\theta_t, n_t)}{\partial n_t}\frac{\partial n_t}{\partial \phi}$$
where
$$\frac{\partial P(\theta_t, n_t)}{\partial n_t} = \Delta u(\theta_t, n_t) \quad \textit{and} \quad \frac{\partial n_t}{\partial \phi} = \delta e^{-\delta(t-\tau)}$$
Therefore, the deviation is not profitable if:
$$\int_{t=\tau}^{\infty} e^{-(\rho+\delta)(t-\tau)}\mathbb{E}\left[\Delta u(\theta_t, n_t)\right] dt \leq 0 \tag{3.2}$$

Now suppose that at a given time τ it is optimal to choose $\phi_\tau > 0$ and we choose a similar deviation, but with $d\phi < 0$. Similar calculations imply that this deviation is not profitable if:
$$\int_{t=\tau}^{\infty} e^{-(\rho+\delta)(t-\tau)}\mathbb{E}\left[\Delta u(\theta_t, n_t)\right] dt \geq 0 \tag{3.3}$$

This implies the following necessary conditions for optimality: if $\phi_\tau = 0$ then (3.2) holds; if $\phi_\tau = 1$ then (3.3) holds; and if $\phi_\tau \in (0, 1)$ then both hold with equality. Notice that those are exactly the necessary (and sufficient) conditions for a Nash Equilibrium for the agents' game with $\sigma > 0$ (see equation (2.1)). That is, when σ is strictly positive, the agents' problem solution is the same as maximizing the Potential function.

□

3.2 *F-P* Planner vs. Benevolent Planner

Notice that, as expected, the *F-P* planner[1] will be different from the benevolent central planner of this game. We will show that, instead of accounting entirely for the externalities as the benevolent social planner does, the *F-P* planner will account just for part of it.

For instance, let $W(\theta, n)$ be the benevolent central planner's instantaneous welfare. Then,

$$W(\theta_t, n_t) = n_t u_1(\theta_t, n_t) + (1 - n_t)u_0(\theta_t, n_t) \tag{3.4}$$

Corollary 3.1. *For simplicity, assume that $u_0(\theta_t, n_t) = 0$. Then, the potential function can be written as*

$$P(\theta_t, n_t) = W(\theta_t, n_t) - \int_0^{n_t} v\frac{\partial u_1(\theta_t, v)}{\partial v}dv$$

Proof. See appendix. □

[1] As we denote the agent solving the problem $\mathcal{P}$

When agents revise their actions from 0 to 1, two types of externalities emerge. The one caused by agents changing from 0 to 1 on those currently in state 1, and the one caused by agents currently in state 1 on agents revising their action to 1.

As one can notice, the *F-P* planner does not account entirely for the externalities present in the game. In fact, it disregards the externalities each additional agent choosing action 1 causes on all agents in state 1.

To see why it accounts only partially for the externalities of the game, assume, for simplicity that $u_0(\theta_t, n_t) = 0$. Then, from (3.4) we can infer that

$$\frac{\partial W(\theta_t, n_t)}{\partial n_t} = u_1(\theta_t, n_t) + n_t \frac{\partial u_1(\theta_t, n_t)}{\partial n_t}$$

where $\frac{\partial u_1(\theta_t, n_t)}{\partial n_t}$ is the externalities generated by n agents currently in state 1 on all agents in state 1. Notice that, the *F-P* planner discounts some of it $\left(\int_0^{n_t} v \frac{\partial u_1(\theta_t, v)}{\partial v} dv\right)$, but not all of it, since

$$sup \left[\int_0^{n_t} v \frac{\partial u_1(\theta_t, v)}{\partial v} dv\right] = n_t^2 \frac{\partial u_1(\theta_t, n_t)}{\partial n_t} < n_t \frac{\partial u_1(\theta_t, n_t)}{\partial n_t}$$

for $n_t \in (0, 1)$.

3.3 A novel proof for the bifurcation probabilities

Burdzy et al. [1998] showed that, as $\mu \to 0$ and $\sigma^2 \to 0$, the economy bifurcates almost instantaneously up or down. Moreover, the probability at which the economy bifurcates up or down, at the threshold, are $(1 - n_t)$ and n_t, respectively.

While their proof rely heavily on stochastic calculus and Brownian Motion apparatus, we show, assuming the economy will bifurcate either up or down when $\sigma^2 \to 0$, the same result regarding the bifurcation probabilities with an alternative proof that makes no use of these tools. This was only possible because of the potential function technology.

Proposition 3.2. *Let the state of the economy at instant 0 be such that agents are indifferent between choosing action 1 (going up) and action 0 (going down). Then, as* $\sigma \to 0$ *either the economy will bifurcate up with probability* $(1 - n_0)$ *or the economy will bifurcate down with probability* (n_0).

Proof. Since we are assuming that $\Delta u(\theta_t, n_t)$ is continuous in both arguments, the Potential function will also be continuous in θ_t and in n_t. Therefore, there is no difference between maximizing (3.1) for the limiting case when $\sigma \to 0$ and the case where $\sigma = 0$. Maximization of (3.1) can be done as if $\sigma = 0$, with a constant θ.

The indifference condition characterizing the maximization of the Potential function is given by

$$\int_{t=0}^{\infty} e^{-\rho t} P(\theta_t, n_t^{\uparrow}) dt = \int_{t=0}^{\infty} e^{-\rho t} P(\theta_t, n_t^{\downarrow}) dt \tag{3.5}$$

where $n_t^{\uparrow} = 1 - (1 - n_0)e^{-\delta t}$ and $n_t^{\downarrow} = n_0 e^{-\delta t}$. That is $n_t^{\uparrow}$ and $n_t^{\downarrow}$ determine the dynamics of n_t for the cases that n_t only goes up or only goes down, respectively.

Maximization of the potential function implies that the economy will either go up forever or down forever. When (3.5) holds, both options maximize (3.1). Now consider the problem of an agent is at the equilibrium threshold – where (3.5) holds. The agent knows that either the economy will go up forever or it will go down forever. But what are the probabilities of each case?

Let's say that the economy will bifurcate down with probability p and up with probability $1-p$. At the threshold, V_τ in (2.1) is zero. Therefore, the indifference condition of an agent in the equilibrium threshold becomes:

$$(1-p)\int_0^{\infty} e^{-(\rho+\delta)t} \Delta u(\theta_t, n_t^{\uparrow}) dt + p \int_0^{\infty} e^{-(\rho+\delta)t} \Delta u(\theta_t, n_t^{\downarrow}) dt = 0$$

where $(1-p)$ and p are the probabilities, attributed by the agent in the equilibrium threshold, that n will bifurcate up or down, respectively.

Applying the change of variables $v = n_t^{\uparrow} = 1 - (1 - n_0)e^{-\delta t}$ and $v = n_t^{\downarrow} = n_0 e^{-\delta t}$ in the integrals above, we get:

$$(1-p)\int_{n_0}^{1} \left(\frac{1-v}{1-n_0}\right)^{\frac{\rho}{\delta}} \frac{\Delta u(\theta_t, v)}{(1-n_0)} dv + p \int_0^{n_0} \left(\frac{v}{n_0}\right)^{\frac{\rho}{\delta}} \frac{\Delta u(\theta_t, v)}{n_0} dv = 0 \tag{3.6}$$

Recall that integrating by parts we have:

$$\int f'(x)g(x)dx = f(x)g(x) - \int g'(x)f(x)dx$$

Thus, from (3.6) and noticing that $\frac{\partial P(\theta_t, v)}{\partial v} = \Delta u(\theta_t, v)$ we have:

$$\begin{aligned} &(1-p)\left\{\left[\frac{-P(\theta_t, n_0)}{(1-n_0)}\right] - \int_{n_0}^{1} \frac{\partial \left[\left(\frac{1-v}{1-n_0}\right)^{\frac{\rho}{\delta}} \frac{1}{(1-n_0)}\right]}{\partial v} P(\theta_t, v) dv\right\} + \\ &+p\left\{\left[\frac{P(\theta_t, n_0)}{n_0}\right] - \int_0^{n_0} \frac{\partial \left[\left(\frac{v}{n_0}\right)^{\frac{\rho}{\delta}} \frac{1}{n_0}\right]}{\partial v} P(\theta_t, v) dv\right\} = 0 \end{aligned} \tag{3.7}$$

Notice that

$$\frac{\partial}{\partial v}\left[\left(\frac{1-v}{1-n_0}\right)^{\frac{\rho}{\delta}} \frac{1}{(1-n_0)}\right] = \frac{-\rho}{\delta(1-n_0)(1-v)} \left(\frac{1-v}{1-n_0}\right)^{\frac{\rho}{\delta}}$$

and that

$$\frac{\partial}{\partial v}\left[\left(\frac{v}{n_0}\right)^{\frac{\rho}{\delta}}\frac{1}{n_0}\right]=\frac{\rho}{\delta n_0 v}\left(\frac{v}{n_0}\right)^{\frac{\rho}{\delta}}$$

Then, from (3.7) and performing the change of variables backwards we have:

$$-(1-p)\frac{P(\theta_t,n_0)}{(1-n_0)}+(1-p)\int_0^\infty\frac{\rho e^{-\rho t}}{(1-n_0)}P(\theta_t,n_t^{\uparrow})dt+p\frac{P(\theta_t,n_0)}{n_0}-p\int_0^\infty\frac{\rho e^{-\rho t}}{n_0}P(\theta_t,n_t^{\downarrow})dt=0$$

Rearranging and multiplying both sides by $n_0(1-n_0)$ we get:

$$n_0(1-p)\left[P(\theta_t,n_0)-\int_0^\infty\rho e^{-\rho t}P(\theta_t,n_t^{\uparrow})dt\right]=(1-n_0)p\left[P(\theta_t,n_0)-\int_0^\infty\rho e^{-\rho t}P(\theta_t,n_t^{\downarrow})dt\right]$$

Therefore, from (3.5) we can conclude that $n_0(1-p)=(1-n_0)p$, or $p=n_0$. That is, n_0 is the probability that n will bifurcate down and $(1-n_0)$ is the probability that n will bifurcate up. □

4 Improving *F-P*'s Tractability

In order to improve the model's tractability, we propose two approaches. The first one involves the potential function described above. That is, since we showed that we can now solve the model by maximizing the utility of a representative agent, we may define it recursively and use the dynamic programming tools appropriately. Secondly, we propose a numerical method that would help us compute the threshold when σ is small relative to δ.

4.1 Recursive Approach

Notice that, proposition 2.1 let us now solve the *F-P* game by maximizing the utility of a single representative agent, without the need to worry about what others are doing. In fact, the representative agent will decide what everybody will do depending on the state of the economy. Therefore, we can define problem $\mathcal{P}$ recursively as

$$V(\theta_t, n_t) = \underset{\phi_t \in [0,1]}{max} \mathbb{E}\left[\int_t^{t+dt} e^{-\rho(s-t)} P(\theta_s, n_s) ds + e^{-\rho dt} V(\theta_{t+dt}, n_{t+dt})\right]$$

subject to

$$d\theta_t = \mu dt + \sigma dZ_t$$
$$dn_t = \delta(\phi_t - n_t)dt$$

which leads, from *Itô's* lemma, to the following HJB equation

$$\rho V(\theta_t, n_t) = \underset{\phi_t \in [0,1]}{max}\left\{P(\theta_t, n_t) + \mu V_\theta + \delta(\phi_t - n_t)V_n + \frac{\sigma^2}{2} V_{\theta\theta}\right\}$$

From Frankel and Pauzner [2000], we know that

$$\phi_t = \begin{cases} 1, & \text{for } \theta \geqslant \theta^*(n) \\ 0, & \text{for} \theta < \theta^*(n) \end{cases}$$

however, we don't know what $\theta^*(n)$ looks like, so we shall define

$$V^H(\theta_t, n_t) := \frac{1}{\rho}\left[P(\theta_t, n_t) + \mu V_\theta^H + \delta(1 - n_t)V_n^H + \frac{\sigma^2}{2} V_{\theta\theta}^H\right] \tag{4.1}$$

and

$$V^L(\theta_t, n_t) := \frac{1}{\rho}\left[P(\theta_t, n_t) + \mu V_\theta^L - \delta n_t V_n^L + \frac{\sigma^2}{2} V_{\theta\theta}^L\right] \tag{4.2}$$

whose general solutions follow, respectively, the inhomogeneous backward heat equations below

$$\frac{2(1-n)^{\frac{\rho}{\delta}}}{\sigma^2}P(\theta,n)+U^H_{\eta_h}+U^H_{z_h z_h}=0 \tag{4.3}$$

and

$$\frac{2n^{\frac{\rho}{\delta}}}{\sigma^2}P(\theta,n)+U^L_{\eta_l}+U^L_{z_l z_l}=0 \tag{4.4}$$

where,

$$\begin{cases} U^H(z_h,\eta_h)=V^H(\theta,n)(1-n)^{\frac{\rho}{\delta}} \\ z_h=\theta+\frac{\mu}{\delta}ln(1-n) \\ \eta_h=-\frac{\sigma^2}{2\delta}ln(1-n) \\ U^L(z_l,\eta_l)=V^L(\theta,n)(n)^{\frac{\rho}{\delta}} \\ z_l=\theta+\frac{\mu}{\delta}ln(n) \\ \eta_l=-\frac{\sigma^2}{2\delta}ln(n) \end{cases}$$

The recursive approach has led us to more familiar grounds. Hence, in order to obtain the threshold $\theta^*(n)$ we need to solve both (4.3) and (4.4) noticing that, at the threshold,

$$V^H(\theta^*(n),n)=V^L(\theta^*(n),n)$$

The search for initial conditions, which would help us properly determine the solutions for the heat equations above, is still subject for future work. Next we discuss how this recursive approach behaves in light of an example.

4.1.1 Linear Utility

Assume, as in Guimares et al. [2017], that agents' relative payoff function is given by

$$\Delta u(\theta_t,n_t)=\theta_t+\gamma n_t$$

where $\gamma>0$. Hence, the potential flow can be written as

$$P(\theta_t,n_t)=\int_0^{n_t}\Delta u(\theta_t,v)dv=\theta_t n_t+\frac{\gamma}{2}n_t^2$$

Following, we show that our recursive approach is in line with the existing results in the literature for the vanishing shocks case, and we also show that the linear case provides us with an interesting interpretation when $\sigma^2>0$.

4.1.1.1 Vanishing shocks case ($\mu = 0$ and $\sigma^2 \to 0$)

First, observe that, by continuity we can write (4.1) and (4.2) as

$$\rho V^H_{\sigma\to 0} = \left(\theta n + \frac{\gamma}{2}n^2\right) + \delta(1-n)V^H_n$$

and

$$\rho V^L_{\sigma\to 0} = \left(\theta n + \frac{\gamma}{2}n^2\right) - \delta n V^L_n$$

Now, notice that $(1-n)^{\frac{\rho}{\delta}}$ and $(n)^{\frac{\rho}{\delta}}$ are, respectively, the integrating factors of the partial differential equations (pdes) above. Hence, one can easily obtain

$$V^H_{\sigma\to 0} = \left[\frac{1}{\delta}\int_n^1 (1-v)^{\frac{\rho}{\delta}-1}\left(\theta v + \frac{\gamma}{2}v^2\right)dv\right](1-n)^{-\frac{\rho}{\delta}} = \frac{\theta}{\rho} - \frac{\theta(1-n)}{\rho+\delta} + \frac{\gamma}{2\rho} - \frac{\gamma(1-n)}{\rho+\delta} + \frac{\gamma(1-n)^2}{2(\rho+2\delta)}$$

and

$$V^L_{\sigma\to 0} = \left[\frac{1}{\delta}\int_0^n (v)^{\frac{\rho}{\delta}-1}\left(\theta v + \frac{\gamma}{2}v^2\right)dv\right](n)^{-\frac{\rho}{\delta}} = \frac{\theta n}{\rho+\delta} + \frac{\gamma n^2}{2(\rho+2\delta)}$$

Moreover, we know that, at the threshold,

$$V^H_{\sigma\to 0}(\theta^*(n), n) = V^L_{\sigma\to 0}(\theta^*(n), n)$$

Therefore, the threshold is represented by

$$\theta^*(n) = -\frac{\gamma\delta}{\rho+2\delta} - \frac{\gamma\rho}{\rho+2\delta}n$$

which is exactly the same result derived in Guimares et al. [2017].

4.1.1.2 General case ($\mu = 0$ and $\sigma^2 > 0$)

We assume, for simplicity, that $\mu = 0$. The real general case wouldn't fix μ, however, as the most interesting parameter is the variance, we are going to deal with this relatively general case where $\mu = 0$ and $\sigma^2 > 0$.

Now, (4.1) and (4.2) can be written respectively as

$$\rho V^H_{\sigma>0} = \theta n + \frac{\gamma}{2}n^2 + \delta(1-n)V^H_n + \frac{\sigma^2}{2}V^H_{\theta\theta}$$

and

$$\rho V^L_{\sigma>0} = \theta n + \frac{\gamma}{2}n^2 - \delta n V^L_n + \frac{\sigma^2}{2}V^L_{\theta\theta}$$

Notice that, the following transformations

$$V^H_{\sigma>0} = W^H + \frac{\theta}{\rho} - \frac{\theta(1-n)}{\rho+\delta} + \frac{\gamma}{2\rho} - \frac{\gamma(1-n)}{\rho+\delta} + \frac{\gamma(1-n)^2}{2(\rho+2\delta)}$$

and

$$V^L_{\sigma>0} = W^L + \frac{\theta n}{\rho+\delta} + \frac{\gamma n^2}{2(\rho+2\delta)}$$

convert the pdes above, respectively, into

$$\rho W^H = \delta(1-n)W^H_n + \frac{\sigma^2}{2}W^H_{\theta\theta}$$

and

$$\rho W^L = -\delta n W^H_n + \frac{\sigma^2}{2}W^L_{\theta\theta}$$

Hence, one can conclude that both value functions can be written as

$$V^H_{\sigma>0} = W^H + V^H_{\sigma\to 0}$$

and

$$V^L_{\sigma>0} = W^L + V^L_{\sigma\to 0}$$

where W^H and W^L are the option values of the nonstandard option present in this game, such that the representative agent can revise his action upon hitting the threshold.

4.2 Hitting Time Distribution Approach

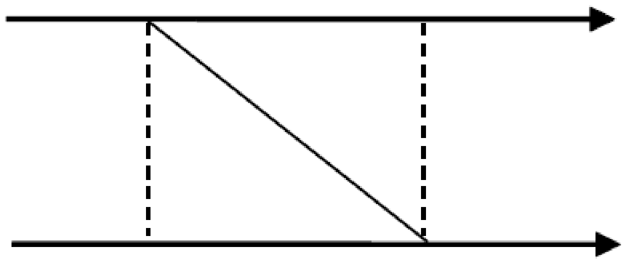
Suppose that σ is small relative to δ. That is, movements in n are much larger than movements in θ. Hence the economy will only cross the threshold when it is close to $n = 0$ and $n = 1$.

Moreover, both $V^H(\theta^*(0), 0)$ and $V^L(\theta^*(1), 1)$ can be written as functions of each other, the first hitting time $H_a := inf\ (t : \theta_t = a)$ given the distance $\Delta_{0,1} := \theta^*(0) - \theta^*(1)$, and the expected value of θ given the distance $\Delta_{0,1}$ and the first hitting time.

Notice that, from Schilling and Partzsch [2014], the first hitting time distribution of a brownian motion with no drift $\mu = 0$ is represented by

$$f_{H_a}(t) = \frac{|\Delta_{0,1}|e^{-\frac{(\Delta_{0,1})^2}{2t}}}{\sqrt{2\pi t^3}}$$

Figure 3 – Small σ Dynamics



Moreover, the expected value of θ given the distance $\Delta_{0,1}$ and the first hitting time could be estimated numerically. Therefore, we could solve numerically for the threshold, where the iteration process would be done at the distance $\Delta_{0,1}$.

One should be attentive that, this numerical method is an approximation for the cases when the variance is small relative to the frictions. However, it is not as restrictive as the limiting cases of vanishing shocks or frictions. Thus we could perform some interesting comparative statics for slight increments in σ, for example.

5 Conclusion

The *F*-*P* model was, up to now, considered to be tractable only for the limiting cases with vanishing shocks or vanishing frictions. This dissertation, however, presented a technique that proves useful for solving the model without the need for small shocks or frictions. Additionally, the theoretical results shown in this dissertation can be used to broaden the scope of applications in the future, which is, so far, somewhat limited.

Besides, we also proposed a numerical method to approximate the threshold when σ is small relative to δ. We argued that this could lead us to interesting comparative statics, improving from the current results with limiting shocks and frictions.

Finally, we showed that we can prove the same result for the bifurcation probabilities from Burdzy et al. [1998] with a much simpler approach, that is, without the use of Brownian Motion and stochastic calculus apparatus.

Bibliography

K. Burdzy, D. M. Frankel, and A. Pauzner. -on the time and direction of stochastic bifurcation. In *Asymptotic Methods in Probability and Statistics*, pages 483–500. Elsevier, 1998.

T. R. Daniëls. Unique equilibrium in a dynamic model of speculative attacks. *De Economist*, 157(4):417–439, 2009.

D. Frankel and A. Pauzner. Resolving indeterminacy in dynamic settings: the role of shocks. *The Quarterly Journal of Economics*, 115(1):285–304, 2000.

D. M. Frankel and K. Burdzy. Shocks and business cycles. *Advances in Theoretical Economics*, 5 (1), 2005.

B. Guimaraes. Dynamics of currency crises with asset market frictions. *Journal of International Economics*, 68(1):141–158, 2006.

B. Guimaraes and C. Machado. Dynamic coordination and the optimal stimulus policies. *The Economic Journal*, 2017.

B. Guimaraes and A. E. Pereira. Qwerty is efficient. *Journal of Economic Theory*, 163:819–825, 2016.

B. Guimares, C. H. Machado, and A. E. Pereira. Dynamic coordination with timing frictions: theory and applications. 2017.

Z. He and W. Xiong. Dynamic debt runs. *The Review of Financial Studies*, 25(6):1799–1843, 2012.

D. Monderer and L. S. Shapley. Potential games. *Games and economic behavior*, 14(1):124–143, 1996.

W. H. Sandholm. Potential games with continuous player sets. *Journal of Economic theory*, 97 (1):81–108, 2001.

R. L. Schilling and L. Partzsch. *Brownian motion: an introduction to stochastic processes.* Walter de Gruyter GmbH & Co KG, 2014.

Appendix

Corollary 0.1.

Proof. It is assumed that $u_0(\theta_t, n_t) = 0$. Then,

$$W(\theta_t, n_t) = n_t u_1(\theta_t, n_t)$$

Notice that, if we integrate by parts, we get

$$\int_0^{n_t} v \frac{\partial u_1(\theta_t, v)}{\partial v} dv = n_t u_1(\theta_t, n_t) - \int_0^{n_t} u_1(\theta_t, v) dv$$

Therefore,

$$W(\theta_t, n_t) - \int_0^{n_t} v \frac{\partial u_1(\theta_t, v)}{\partial v} dv = \int_0^{n_t} u_1(\theta_t, v) dv$$

which is exactly $P(\theta_t, n_t)$ when $u_0(\theta_t, n_t) = 0$.

□