FUNDAÇÃO GETÚLIO VARGAS

ESCOLA DE ECONOMIA DE SÃO PAULO

**BRUNO BARSANETTI**

# Signaling in Dynamic Markets with Adverse Selection

São Paulo
2014

FUNDAÇÃO GETÚLIO VARGAS

ESCOLA DE ECONOMIA DE SÃO PAULO

**BRUNO BARSANETTI**

# Signaling in Dynamic Markets with Adverse Selection

Dissertação apresentada à Escola de Economia de São Paulo da Fundação Getúlio Vargas, como requisito para a obtenção do título de Mestre em Economia de Empresas

Área do conhecimento: Microeconomia

Orientador: Braz Camargo

São Paulo
2014

Barsanetti, Bruno.
Signaling in Dynamic Markets with Adverse Selection / Bruno Barsanetti. - 2014.
28 f.

Orientador: Braz Camargo
Dissertação (mestrado) - Escola de Economia de São Paulo.

1. Seleção adversa. 2. Preços. 3. Equilíbrio econômico. 4. Microeconomia. I. Camargo, Braz. II. Dissertação (mestrado) - Escola de Economia de São Paulo. III. Título.

CDU 330.101.52

Bruno Barsanetti

# Signaling in Dynamic Markets with Adverse Selection

Dissertação apresentada à Escola de Economia de São Paulo da Fundação Getúlio Vargas, como requisito para a obtenção do título de Mestre em Economia de Empresas

Área do conhecimento: Microeconomia

Orientador: Braz Camargo

**Data de aprovação:**

20/03/2014

**Banca Examinadora:**

Prof. Dr. Braz Camargo
(Orientador)
EESP/FGV

Prof. Dr. Victor Filipe Martins-da-Rocha
EESP/FGV

Prof. Dr. Humberto Moreira
EPGE/FGV

# AGRADECIMENTOS

Agradeço ao Braz Camargo, pela valiosa orientação. Sou também grato a participantes do seminário de tese na EESP, por comentários sobre a incipiente idéia inicial para essa dissertação. Da mesma forma, agradeço à banca de qualificação.

Aos meus colegas na EESP, com quem passei dois ótimos anos em um ambiente estimulante.

No plano pessoal, agradeço a meus pais e irmã por todo o apoio incondicional, não só durante o mestrado, mas em toda minha vida. Agradeço à Simone, por me incentivar e inspirar.

Por fim, agradeço à Capes pela ajuda financeira durante os meus estudos de mestrado, que resultaram na presente pesquisa.

# ABSTRACT

We consider trade in dynamic decentralized markets with adverse selection. Unlike the literature so far, we assume that informed sellers (and not uninformed buyers) make take-it-or-leave-it offers, so that signaling through prices is possible. We establish a partial characterization of the equilibrium set, provide necessary and sufficient conditions for the existence of an equilibrium, and show that all equilibria involve signaling if the adverse selection problem is severe enough. Moreover, we prove the somewhat surprising result that the highest welfare achieved in equilibrium is invariant to market frictions. We also provide a necessary and sufficient condition for the existence of separating equilibria, completely characterize such equilibria, and show that the set of equilibrium payoffs for separating equilibria is invariant to market frictions. We conclude with a complete characterization of the equilibrium set in the two-type case, and compare our results to the those in Moreno and Wooders (2010), who analyze the case in which buyers have all the bargaining power. Our results show that signaling through prices can have a non-trivial impact on market outcomes and welfare.

Key-words: adverse selection, decentralized markets, signaling

# RESUMO

Nesta dissertação, consideram-se trocas em mercados descentralizados com seleção adversa. Diferentemente da literatura até o momento, supomos que vendedores informados (e não compradores desinformados) fazem ofertas *take-it-or-leave-it*, de forma que sinalização através de preços é possível. Estabelecemos uma caracterização parcial do conjunto de equilíbrio, encontramos condições necessárias e suficientes para a existência de um equilíbrio e mostramos que todo equilíbrio apresenta sinalização se o problema de seleção adversa for suficientemente severo. Além disso, provamos o resultado surpreendente que o maior bem-estar atingido em equilíbrio é invariante às fricções do mercado. Também apresentamos condições necessárias e suficientes para a existência de equilíbrios separantes, que caracterizamos completamente. Mostramos que o conjunto de *payoffs* associados a equilíbrios separantes é invariante às fricções. Concluímos com uma caracterização completa do conjunto de equilíbrio com apenas dois tipos, e comparamos nossos resultados com os de Moreno e Wooders (2010), que analisam o caso em que compradores têm todo o poder de mercado. Nossos resultados mostram que sinalização através dos preços tem um impacto não trivial tanto nos resultados do mercado quanto no bem-estar.



Palavras-chave: seleção adversa, mercados descentralizados, sinalização

## CONTENTS

# 1 INTRODUCTION

There is by now a large literature on dynamic markets with adverse selection. In this paper, we contribute to this literature by analyzing signaling through prices in dynamic decentralized markets with adverse selection.[1] The possibility of signaling through prices seems natural in those markets, and one would like to know what is the impact of such form of signaling on market outcomes.

We consider decentralized markets in which there is a constant inflow of buyers and sellers in each period. Each seller owns one unit of a durable and indivisible good. The good can be of one of finitely many different qualities, and each seller is privately informed about the type of her good. The types of the goods are ordered in the sense that the value of a good to buyers and sellers is increasing in the good's type (i.e., the higher the type of the good, the more valuable the good is to both buyers and sellers). In each period, the buyers and sellers in the market are randomly and anonymously matched in pairs. Once matched to a buyer, a seller makes a take-it-or-leave-it offer to the buyer. If the buyer accepts the offer, then trade takes place and both agents leave the market. Otherwise, the buyer and the seller stay in the market.

Our environment is similar to the environment in Moreno and Wooders (2010) but with two important differences. First, there are finitely many types of sellers, as opposed to only two, and we do not place any a priori restrictions on gains from trade; in particular, gains from trade can be negative for one or more types of goods. Second, sellers, and not buyers, have all the bargaining power. We consider symmetric and stationary perfect Bayesian equilibria.

We begin by establishing some basic properties of equilibria and by providing necessary and sufficient conditions for the existence of an equilibrium. We also show that if the adverse selection problem is severe, which is the natural assumption to make, then all equilibria involve signaling through prices in the sense that sellers with a good of sufficiently high quality will never offer the lowest price that is offered by sellers with the good of lowest quality.[2] Thus, when the adverse selection problem is severe, a good of sufficiently high quality can only trade with delay, otherwise the incentive compatibility constraint for the sellers with the good of lowest quality is violated (i.e, such sellers could profitably deviate by charging a higher price).

We then analyze how welfare depends on market frictions. It is intuitive that if a certain type of good is traded with delay in equilibrium, then this delay increases as the time interval between two consecutive trading opportunities diminishes, as a reduction in market frictions relaxes incentive compatibility constraints. Indeed, we show that if the adverse selection problem is

[1] So far, the literature on dynamic markets with adverse selection has abstracted from signaling through prices by either considering trade in environments in which the uninformed agents make offers (see, e.g., Camargo and Lester (2013), Chiu and Koeppl (2011), Daley and Green (2012), Hörner and Vieille (2009), Kaya and Kim (2014), and Kim (2012)) or by considering trade in Walrasian markets (see, e.g., Fuchs and Skrzypacz (2014), Guerrieri and Shimer (2013), Janssen and Roy (2002), and Kurlat (2013)).

[2] The adverse selection problem is severe when the price that makes a seller with a good of the highest quality indifferent between keeping the good and selling it is greater than the expected quality to buyers of the goods entering the market.

severe, then the probability that a good transacts in each period converges to zero as market frictions disappear if the quality of the good is high enough. This result suggests that welfare can decrease as market frictions vanish. As it turns out, this is not the case. We prove the somewhat surprising result that the highest welfare possible in equilibrium *is invariant to the level of market frictions*. This result stands is sharp contrast to the result in Moreno and Wooders (2010) that when buyers have all the bargaining power, welfare decreases as market frictions vanish.

Separating equilibria are a natural class of equilibria to analyze, as in these equilibria prices are a complete signal of quality. We provide a necessary and sufficient condition for the existence of a separating equilibrium and completely characterize such equilibria. A consequence of our characterization of separating equilibria is that *the set of equilibrium payoffs for separating equilibria is invariant to market frictions.* We also show, by means of an example, that the least cost separating equilibrium need not be the most efficient equilibrium.

In order to have a better comparison between our results and the results in Moreno and Wooders (2010), we finish our analysis by providing a complete characterization of the equilibrium set when, as in Moreno and Wooders (2010), there are only two types of seller, a seller with a low quality good and a seller with a high quality good, gains from trade are always positive, and they increase with the quality of the good. In this case, we are able to show that: (*i*) equilibrium payoffs are invariant to market frictions, so that welfare does not depend on market frictions as well; and (*ii*) that as frictions vanish the most efficient equilibrium when sellers make take-it-or-leave-it offers is more efficient than the unique equilibrium when buyers make take-it-or-leave-it offers. These results show that signaling through prices can have a non-trivial impact on market outcomes and welfare.

The rest of the paper is organized as follows. We discuss the related literature in the remainder of the Introduction. We describe our environment and define equilibria in Section 2. Section 3 contains the main results of the paper, namely, a partial characterization of the equilibrium set, a necessary and sufficient condition for existence of an equilibrium, the result that all equilibria involve signaling through prices when adverse selection is severe, and the result that the highest welfare possible in equilibrium is invariant to market frictions. We analyze separating equilibria in Section 4, and discuss the two-type case in Section 5. Section 6 concludes. The proofs of all results and omitted details are in the Appendix.

**RELATED LITERATURE**

The idea that prices can be used to signal quality is old. Bagwell and Riordan (1991) is a seminal reference. The environment we consider is a dynamic version of the static environment analyzed in Ellingsen (1997). The paper most closely related to ours is Moreno and Wooders (2010), who also consider stationary equilibria in a dynamic market with adverse selection and a constant inflow of buyers and sellers. Besides considering a more restrictive two-type setting, Moreno and Wooders assume that the uninformed buyers, instead of the informed sellers, set prices. Therefore, the steady-state composition of sellers' types remains endogenous, but prices do not have a signaling function anymore. As frictions vanish, no gains from trade of high-quality goods are realized. This is in sharp contrast to our result that payoffs are not affected by frictions. The transmission of information through prices is the main cause of these divergent conclusions.[3]

## 2 ENVIRONMENT AND EQUILIBRIA

Time is discrete and infinite. In each period, a measure one of sellers and a measure one of uninformed buyers enter a market for an indivisible good. Each seller has private information about the type $i$ of her good, $i \in I = \{1, ..., n\}$.[4] Let $f^i > 0$ be the fraction of type $i$ sellers in the entering mass. Buyers' valuation of a type $i$ good is $v^i \geq 0$, while sellers' valuation is $c^i \geq 0$. We assume $c^i$ is increasing in $i$, while $v^i$ is non-decreasing. Our environment is quasi-linear: if a type $i$ seller sells to a buyer at a price $t$, their payoffs are respectively $t - c^i$ and $v^i - t$.

In each period, buyers and sellers are randomly and anonymously matched in pairs. Once matched, the seller sets a price $t \in T \equiv \Re_+$. The buyer observes the price and decides whether to trade. If there is trade, both agents leave the market. Otherwise, they remain in it and are randomly matched to other agents in the subsequent period. Time is discounted by a common factor $\delta \in [0, 1)$, that is also a measure of frictions in this market.

We consider stationary and symmetric equilibria.[5] In this case, type $i$ sellers' strategies are described by a cumulative distribution function of offers $G^i : T \to [0, 1]$. Let $S^i$ be the set of offers made with positive probability by type $i$ sellers. The strategy of buyers consists of a function $\sigma : T \to [0, 1]$, such that $\sigma(t)$ denotes the probability an offer $t$ is accepted. Finally, beliefs are characterized by a function $\pi : T \to \Delta(I)$ that associates every $t \in T$ with a probability distribution over types $\{\pi^i(t)\}_{i=1}^n$.

In any given moment, buyers' and type $i$ sellers' payoffs are $V$ and $U^i$, respectively. The reservation price of type $i$ sellers is defined as $r^i \equiv c^i + \delta U^i$, while the reservation price of buyers for the good sold by that type of seller is $w^i \equiv v^i - \delta V$.

Let $M$ and $N^i$ be, respectively, the mass of buyers and of type $i$ sellers in the steady-state. We denote the steady-state distribution of types by $\{\tilde{f}^i\}_{i=1}^n$: for every $i \in I$, $\tilde{f}^i \equiv \frac{N^i}{\sum_{j=1}^n N^j}$.

[3]The result that offers from the informed party may lead to more efficient outcomes than in the case all offers are made by the uninformed party resembles a result in bilateral bargaining with incomplete information and common values: Gerardi, Horner and Maestri (2013) show that the most efficient equilibrium of a bargaining game in which the informed party makes all offers always achieve a better result than the equilibrium of the bargaining game in which the uninformed party makes all offers, as in Deneckere and Liang (2006).

[4]For convenience, we assign types to both goods and sellers: a seller of type $i$ is one who owns a good of type $i$.

[5]Since there is a continuum of agents, the assumption of symmetry is not restrictive: an asymmetric equilibrium is equivalent to a symmetric one. For some arguments in the appendix, we consider asymmetric equilibria in which sellers' strategies are pure. In this case, we redefine types such that the equilibrium is symmetric in the new type space. Whenever we do this, we denote types by $J$ and their distribution in the entering mass by $\{\varphi^j\}_{j \in J}$.

**Definition 1.** *An equilibrium is a list* $\{\sigma, (G^i)_{i=1}^n, (\pi^i)_{i=1}^n, M, (N^i)_{i=1}^n, V, (U^i)_{i=1}^n\}$ *that satisfies:*

**(i)** ***Buyers' strategies are optimal given beliefs:*** *for all* $t \in T$,

$$\sigma(t) \in arg\, max_{\tilde{\sigma}\in[0,1]} \quad \tilde{\sigma}(\sum_{i=1}^{n} \pi^i(t)w^i - t)$$

**(ii)** ***Sellers' strategies are optimal:*** *for every* $i \in I$,

$$t \in S^i \quad \Rightarrow \sigma(t)(t - r^i) \geq \sigma(t')(t' - r^i) \quad \forall t' \in T$$

**(iii)** ***Beliefs are rational:*** *if* $t \in \cup_{i=1}^n S^i$, *then* $\{\pi^i(t)\}_{i=1}^n$ *follows Bayes' rule.*

**(iv)** ***Buyers' value function:***

$$V = \sum_{i=1}^{n} \tilde{f}^i \int_T max\{\sum_{i=1}^{n} \pi^i(t)w^i - t, 0\}\, dG^i(t) + \delta V$$

**(v)** ***Sellers' value functions:*** *for each type* $i \in I$,

$$U^i = \int_T \sigma(t)(t - r^i)\, dG^i(t) + \delta U^i$$

**(vi)** ***Stationarity:***

$$M(\sum_{i=1}^{n} \tilde{f}^i \int_T \sigma(t) dG^i(t)) = 1$$

*and, for every* $i \in I$,

$$N^i \int_T \sigma(t) dG^i(t) = f^i$$

When a type $i$ seller meets a buyer, they bargain over a surplus of $w^i - r^i$. Condition (i) requires that buyers only accept offers that give them some expected surplus, considering beliefs. Condition (ii) implies that prices offered with positive probability by type $i$ sellers maximize the expected surplus extracted. Conditions (iv) and (v) relate, respectively, buyers' and sellers'payoffs to the expected surplus they get in each period. Finally, the stationarity condition (vi) requires that, for each type of agent, the mass leaving the market equals the entering mass.

Notice that we impose no restrictions on beliefs off the equilibrium path. It is possible to show, as in Ellingsen (1997), that the intuitive criterion has no bite in our environment and that D1 eliminates all but the separating equilibria; the details are available upon request. We do not find D1 very compelling since it eliminates the most natural equilibrium when adverse selection is not severe (see Definition 2 below), namely, the pooling equilibrium in which all sellers offer the price $t = \sum_{i=1}^{n} f_i v_i$ and the buyers accept this price.

# 3 CHARACTERIZATION AND EXISTENCE

In this section, we present some results that characterize basic properties of any given equilibrium. We then discuss conditions for the existence of an equilibrium. Finally, we discuss signaling through prices.

### BASIC RESULTS

Since buyers may always reject offers, $V \geq 0$ and $w^i \leq v^i$ for every $i \in I$. A type $i$ seller, on the other hand, may always offer $c^i$, which implies $U^i \geq 0$ for every type $i$. Hence, the reservation price of a given type of seller is at least her valuatiom, that is, $r^i \geq c^i$. There are other restrictions on sellers payoffs. If an offer is sufficiently small, it will be accepted regardless of beliefs. In particular, since $\sum_{i=1}^{n} \pi^i(t) w^i \geq w^1$ for every offer $t$, then $t < w^1$ implies that $\sigma(t) = 1$. Hence, $U^i \geq w^1 - c^i$ for every $i \in I$.

**Lemma 1.**

**(i)** *For any $i \in I$, if $t \in S^i$ and $t < r^i$, then $\sigma(t) = 0$.*

**(ii)** $U^i > 0$ *for every $i < n$.*

Item (i) of Lemma 1 just states that an offer below the reservation price is made by a seller only if she expects it to be rejected. Item (ii) states that sellers of lower types must get some surplus in every period, since stationarity requires that trade of higher quality goods occurs with positive probability. A direct consequence of the previous lemma is that no type lower than $n$ makes offers that are rejected with probability 1.

The following result shows that both a seller's reservation price and her offers are increasing in her type. Her payoff, on the other hand, is decreasing in $i$

**Lemma 2.** *If $i > j$, then $r^i > r^j$, $inf\ S^i \geq sup\ S^j$ and $U^i < U^j$.*

It follows directly from Lemma 2 that, for any $j < k < i$ such that $S^i \cap S^j = \{t\}$, $S^k = \{t\}$. That is, pooling between two types necessarily leads to pooling with any intermediate type.

Since only the highest price in $S^i$ may be offered by sellers of types higher than $i$ and only the lowest price in $S^i$ may be offered by types lower than $i$, we show in the appendix that, on the equilibrium path, there is a finite number of offers that are accepted with positive probability. Denote these offers by $t_1 < t_2 < ... < t_m$. For each $1 \leq k \leq m$, let $I_k$ be the set of types that offer $t_k$, i.e., $I_k \equiv \{i \in I : t_k \in S^i\}$. From Lemma 2, $I_1 \leq I_2 \leq ... \leq I_m$. In equilibrium, no seller type has incentive to deviate to any other offer in the equilibrium path, which implies the following restrictions:

$$\sigma(t_k)(t_k - r^i) \geq \sigma(t_h)(t_h - r^i) \qquad \forall\, i \in I_k\,, \forall\, k, h \in \{1, ..., m\} \tag{1}$$

For each $k$, let $\underline{i}(k)$ and $\bar{i}(k)$ be respectively the lowest and the highest type in $I_k$. The lemma below shows that, from all restrictions in (1), only local ones are relevant. That is, if the lowest type that makes a given offer $t_k$ has no incentive to offer $t_{k-1}$ and the highest type that offers $t_k$ has no incentive to offer $t_{k+1}$, then there are no profitable deviations on the equilibrium path.

**Lemma 3.** *If, for every* $1 \leq k < m$*,*

$$\sigma(t_k)(t_k - r^{\bar{i}(k)}) \geq \sigma(t_{k+1})(t_{k+1} - r^{\bar{i}(k)}) \text{ and } \sigma(t_{k+1})(t_k - r^{\underline{i}(k+1)}) \geq \sigma(t_k)(t_k - r^{\underline{i}(k+1)})$$

*Then (1) is satisfied.*

Lemma 3 identifies conditions that rule out profitable deviations on the equilibrium path. It remains to discuss deviations off the equilibrium path. However these are not important, since we may always choose beliefs off the equilibrium path such that all relevant deviations belong to the set of offers made with positive probability. The intuition is that any offer $t \notin \cup_{i=1}^n S^i$ is either too low ($t < w^1$) to be a profitable deviation or, otherwise, it is rejected for sufficiently pessimistic beliefs.[6]

## EXISTENCE

So far we have characterized the set of equilibria. We now use this characterization to consider the existence of an equilibrium. Given a type $i$, we say there is a gap for $i$ if there are positive gains from trade, that is, if $v^i > c^i$. Let $I_0 \equiv \{i \in I : v^i \leq c^i\}$ denote the set of types with no gap. Given a pair of types $(k, j)$, $k \leq j$, we define the distribution of types $i$ conditional to $k \leq i \leq j$ as a list $\{f^i_{(k,j)}\}_{i=1}^n$ such that $f^i_{(k,j)} \equiv \frac{f^i}{\sum_{l=k}^j f^l}$, if $k \leq i \leq j$, and $f^i_{(k,j)} \equiv 0$, otherwise.

**Proposition 1.** *The following are sufficient and necessary conditions for the existence of an equilibrium:*

1. *For every* $k \in I_0 \setminus \{n\}$ *there is* $j > k$ *such that* $\sum_{i=1}^n f^i_{(k,j)} v^i \geq c^j$.
2. $v^n \geq c^n$

In order to understand the result, notice that condition (1) implies that, for every type $i$ for which there are no gains from trade, there is a higher seller type with which a type $i$ seller could pool one of her offers in such a way that buyers would be willing to accept it. If (1) is

[6]Formally, we may always set, for any $t \notin \cup_{i=1}^n S^i$ such that $t \geq w^1$, $\pi^1(t) = 1$. In this case, $\sigma(t) = 0$ is a best response of buyers to these beliefs. For $t < w^1$, $\sigma(t) = 1$ but no type $i$ is willing to offer it as long as $U^i \geq w^1 - c^i$, a condition that always holds in equilibrium.

violated, every offer from some $i \in I_0 \setminus \{n\}$ would necessarily be rejected. Hence, $U^i = 0$. From Lemma 1, this is incompatible with an equilibrium, since $i < n$. Condition (2), on the other hand, requires that there are no losses from trade with the highest type. If this was not the case, none of the offers from type $n$ sellers would be accepted by buyers, which contradicts stationarity.

The proof that conditions (1) and (2) are sufficient for the existence of an equilibrium is constructive. It is worth mentioning that the constructed equilibrium satisfies $V = 0$ and that sellers' strategies are pure. Therefore, our result is even stronger: the conditions in Proposition 1 actually are sufficient for the existence of an equilibrium with these properties.

Notice conditions (1) and (2) are rather weak requirements that are trivially satisfied if there is a gap for every type. For the remainder of the paper, we assume they are always satisfied.

## SIGNALING THROUGH PRICES

From Lemma 2, prices are non-decreasing in types. Therefore, if there is price dispersion, with more than one price being offered in equilibrium, then prices are a signal of sellers' types. In this subsection, we identify a sufficient condition for price dispersion in equilibrium.

**Definition 2.** *Adverse selection is severe if* $\sum_{i=1}^{n} f^i v^i < c^n$.

Notice that if adverse selection is not severe, that is, if $\sum_{i=1}^{n} f^i v^i \geq c^n$, then there is an equilibrium in which every seller offers the same price, say $t = \sum_{i=1}^{n} f^i v^i$. In this case, the expected value of the good to the buyer is sufficient high to be offered by the highest type. However, when adverse selection is severe, our next result asserts that there is necessarily some price dispersion in equilibrium.

Let $i^* \equiv min\{i \in I : \sum_{k=1}^{n} f_{(1,j)}^k v^k < c^j \; \forall j \geq i\}$. If adverse selection is severe, then there is such $i^*$ that satisfies $1 < i^* \leq n$.

**Proposition 2.** *If adverse selection is severe, then there is* $t \in S^1$ *such that* $t < S^j$ *for any* $j \geq i^*$.

A direct consequence of Proposition 2 is that offers from high types of sellers are necessarily rejected with positive probability, otherwise type 1 sellers would offer them. This implies that sellers of these types face some expected delay before they trade. The corollary below states a stronger result: as frictions vanish, rejection becomes more likely, to the point its probability converges to 1.

**Corollary 1.** *Suppose adverse selection is severe. Consider* $\delta_k \to 1$ *and an associated sequence of equilibria. For each* $k \in \mathbb{N}$ *and each* $j \geq i^*$, *let* $t_k^j \in S_k^j$. *Hence,* $\sigma_k(t_k^j) \to 0$.

The fact that trade probabilities for higher types converge to zero as frictions vanish does not imply that the payoff of these types necessarily goes to zero. Actually, this is not the case: if we interpret frictions as the time gap between successive rounds of trade, then as frictions vanish the expected number of rounds before trade increases without bounds, but the expected time to trade remains finite. We discuss the relation of payoffs to $\delta$ in what follows.

For a given $\delta$, let $\mathbb{U}_0(\delta)$ be the set of payoffs associated to equilibria in which buyers get no surplus, that is, $V = 0$. From the constructive proof of Proposition 1, we know that this set is not empty. Our next result shows that $\mathbb{U}_0(\delta)$ is invariant to $\delta$:

**Lemma 4.** *For every pair* $(\delta, \delta') \in [0,1)^2$, $\mathbb{U}_0(\delta) = \mathbb{U}_0(\delta') \equiv \mathbb{U}_0$.

There may also be equilibria with $V > 0$. Although there is no result similar to Lemma 4 for these equilibria, we may compare welfare $W = V + \sum_{i=1}^{n} f^i U^i$ across these two classes of equilibria. In the appendix, we show that $W \leq \sum_{i=1}^{n} f^i \int_T \frac{\sigma(t)(v^i - c^i)}{1-\delta(1-\sigma(t))} dG^i(t)$, with equality if $V = 0$. Since $V > 0$ implies that buyers' reservation prices are lower, then equilibrium prices are in general also lower. In the appendix, we show that this implies that, for every equilibrium with $V > 0$, there is an equilibrium with $V = 0$ that achieves a higher welfare. Let $\mathbb{W}(\delta)$ be the set of equilibrium welfare associated to $\delta$. Therefore, as a consequence of Lemma 4, the highest welfare achieved by some equilibrium, $\overline{W}(\delta) = max\ \mathbb{W}(\delta)$, is invariant to $\delta$.

**Proposition 3.** *For every pair* $(\delta, \delta') \in [0,1)^2$, $\overline{W}(\delta) = \overline{W}(\delta') \equiv \overline{W}$.

# 4 SEPARATING EQUILIBRIA

We have shown that, if adverse selection is severe, then prices signal quality. We now consider the extent they do so. This leads us naturally to the study of separating equilibria, in which prices fully reveal quality.

We say that an equilibrium is quasi-separating if $v^i \neq v^j$ implies that $S^i \cap S^j = \emptyset$. In this case, prices are a perfect sign of value to buyers, that is, they signal all information that is payoff relevant to them. If $S^i \cap S^j = \emptyset$ for every $i \neq j$, then the equilibrium is separating. Notice that every quasi-separating equilibrium is indeed separating in the case $v^i$ is strictly increasing in $i$.

The following proposition characterize quasi-separating equilibria and establishes their existence:

**Proposition 4.** *There is a quasi-separating equilibrium if, and only if,* $I_0 \backslash \{n\} = \emptyset$ *and* $v^n \geq c^n$.

*Furthermore, any quasi-separating equilibrium satisfies:*

**(i)** $v^n \in S^n$ *and, for every type* $i < n$, $S^i = \{v^i\}$.

**(ii)** $\sigma(v^1) = 1$ *and, for every* $i > 1$, $\sigma(v^i)$ *satisfies* $\sigma(v^i) > 0$ *and the recursive condition:*

$$\frac{v^{i-1} - r^i}{v^i - r^i} \sigma(v^{i-1}) \leq \sigma(v^i) \leq \frac{v^{i-1} - r^{i-1}}{v^i - r^{i-1}} \sigma(v^{i-1}) \tag{2}$$

**(iii)** $V = 0$.

In order to understand Proposition 4, we know that buyers only get some fraction of the surplus if they accept the offer with certainty. This only happens to the lowest offer, that, by Lemma 2, is necessarily made by the lowest type. Since any offer below $w^1$ is accepted with probability one, $w^1$ is precisely the offer that type 1 makes. Therefore, there is no offer that gives some surplus to buyers, and then $V = 0$. This implies that buyers are indifferent between accepting or rejecting any offer on the equilibrium path, such that in a quasi-separating equilibrium we necessarily have that each seller offers as price the value (to buyers) of the good she owns.

Buyers accept these offers with a probability that discourages sellers from deviating and offering a different $v^i$. From Lemma 3, only deviations from types immediately below or immediately above are relevant. This defines, for each $\sigma(v^i)$, the interval described by item (ii).

Finally, notice that the conditions required for the existence of a quasi-separating equilibrium are somewhat stronger than those stated in Proposition 1: now there must be a gap for all types except for type $n$. If this was not the case, then some types would never have their offers accepted, violating stationarity.

The conditions for the existence of a separating equilibrium are slightly stronger:

**Corollary 2.** *There is a separating equilibrium if, and only if, $I_0 \setminus \{n\} = \emptyset$, $v^n \geq c^n$ and $v^i$ is strictly increasing in $i$.*

We now discuss payoffs associated to quasi-separating equilibria. Let $\mathbb{U}^S(\delta)$ be the set of quasi-separating equilibrium payoffs for a given $\delta$. Our next result establishes that this set is invariant to frictions:

**Corollary 3.** *For every pair $(\delta, \delta') \in [0,1)^2$, $\mathbb{U}^S(\delta) = \mathbb{U}^S(\delta') \equiv \mathbb{U}^S$.*

In order to understand the corollary above, notice we may rewrite the recursive condition in item (ii) of Proposition 4 replacing $\{\sigma(v^i)\}_{i=1}^n$ for $\{\kappa(v^i)\}_{i=1}^n$, where $\kappa(t) \equiv \frac{\sigma(t)}{1-\delta(1-\sigma(t))}$. Notice that $\kappa(v^1) = 1$. Some algebra shows that the recursive condition is equivalent to:

$$\frac{v^{i-1} - c^i}{v^i - c^i}\kappa(v^{i-1}) \leq \kappa(v^i) \leq \frac{v^{i-1} - c^{i-1}}{v^i - c^{i-1}}\kappa(v^{i-1}) \quad i = 2, ..., n \tag{3}$$

Since the condition above is independent of $\delta$ and $U^i = \kappa(v^i)(v^i - c^i)$ for every $i \in I$, payoffs are necessarily invariant to frictions.

We may also rank quasi-separating equilibria according to the Pareto criterion. In any quasi-separating equilibrium buyers get no surplus ($V = 0$), so it is necessary to consider only sellers' payoffs. Since the payoff of a type $i$ seller is increasing in $\kappa(v^i)$, the corollary below is a direct consequence of the recursive condition (3):

**Corollary 4.** *A quasi-separating equilibrium that satisfies $\sigma(v^1) = 1$ and $\sigma(v^i) = \frac{v^{i-1}-r^{i-1}}{v^i-r^{i-1}}\sigma(v^{i-1})$ for every $i > 1$ Pareto dominates any other quasi-separating equilibrium.*

We say the quasi-separating equilibrium described in Corollary 4 is a Riley equilibrium. Although it is efficient among the equilibria of its class, for some parameters it may be Pareto dominated by some other equilibrium. We show it by considering a numerical example with three types. Let $c^1 = 0$, $c^2 = 1$ and $c^3 = 4$. Values to buyers are $v^1 = 1$, $v^2 = 2$ and $v^3 = 5$. The distribution of types is $(f^1, f^2, f^3) = (\frac{1}{20}, \frac{9}{20}, \frac{1}{2})$. Fix $\delta = \frac{1}{2}$. Payoffs in the most efficient separating equilibrium are: $V = 0$, $U^1 = 1$, $U^2 = \frac{1}{2}$ and $U^3 = \frac{1}{8}$. Consider now an alternative equilibrium in which sellers of types 1 and 2 are pooled, offering $t = f^1_{(1,2)} v^1 + f^2_{(1,2)} v^2 = \frac{19}{10}$, while type 3 sellers offer $v^3$. Let $\sigma(t) = 1$. If type 2 sellers are indifferent between offering $t$ and $v^3$, then $\sigma(v^3) = \frac{9}{71}$. In this equilibrium, payoffs are: $V = 0$, $U^1 = \frac{19}{20}$, $U^2 = \frac{9}{10}$ and $U^3 = \frac{9}{40}$. Hence, this equilibrium Pareto dominates the Riley equilibrium. We summarize this numerical example in the following table:

| Riley Equilibrium | | | | | |
|---|---|---|---|---|---|
| $\sigma(v^1)$ | $1$ | $U^1$ | $1$ | $r^1$ | $\frac{1}{2}$ |
| $\sigma(v^2)$ | $\frac{1}{3}$ | $U^2$ | $\frac{1}{2}$ | $r^2$ | $\frac{5}{4}$ |
| $\sigma(v^3)$ | $\frac{1}{15}$ | $U^3$ | $\frac{1}{8}$ | $r^3$ | $\frac{33}{32}$ |
| Pooling between Types 1 and 2 | | | | | |
| $\sigma(t)$ | $1$ | $U^1$ | $\frac{19}{10}$ | $r^1$ | $\frac{19}{20}$ |
| | | $U^2$ | $\frac{9}{10}$ | $r^2$ | $\frac{29}{40}$ |
| $\sigma(v^3)$ | $\frac{9}{71}$ | $U^3$ | $\frac{9}{40}$ | $r^3$ | $\frac{329}{80}$ |

Notice that the second equilibrium dominates the first one because separating type 1 sellers is costly, since their proportion in the entering population is too low. In this case, the common price $t$ that sellers of type 1 and 2 offer when they are pooled is only slightly below $v^2$. However, $t$ is accepted with certainty, while $v^2$ is accepted in the Riley equilibrium with probability $\frac{1}{3}$. Hence, both type 1 and type 2 sellers gain by pooling. Furthermore, $v^3$ is now accepted at a highest probability, which also improves type 3 sellers' payoff.

Hence, although their characterization is intuitive, quasi-separating equilibria are not necessarily the most plausible ones, since they may be Pareto dominated. We also remark that many of their properties cannot be generalized to the whole set of equilibria. For example, pooling among the lowest types could lead to an equilibrium in which $\sum_{i=1}^{n} \pi^i(t) v^i > t$ for the lowest offer $t$. In this case, $V > 0$.

# 5 TWO TYPE EXAMPLE

In this section, we illustrate some of the points of this paper by analyzing environments with only two types. In this case, there is a simple characterization of the set of equilibria, that is also completely ranked according to the Pareto criterion. This is also the environment analyzed by Moreno and Wooders (2010).

Consider $I = \{1, 2\}$. In this way, $i = 1$ denotes low-quality sellers while $i = 2$ denotes high-quality ones. We assume that $c^1 < v^1 < c^2 \leq v^2$. In this example, we characterize distributions over types by the probability assigned to high-quality sellers: let $f \equiv f^2$, $\tilde{f} \equiv \tilde{f}^2$ and, for every $t \in [0, v^2]$, $\pi(t) \equiv \pi^2(t)$. We assume that the fraction of high-quality types is sufficiently low, such that adverse selection is severe:

$$f < \hat{f} \equiv \frac{c^2 - v^1}{v^2 - v^1} \tag{4}$$

From Proposition 2, there is $t \in S^1$ such that $t < S^2$. We claim that $t = w^1$. Since $\pi(t) = 0$, $t > w^1$ would imply that $\sigma(t) = 0$, violating Lemma 1. If $t < w^1$, then $\sigma(t + \epsilon) = 1$ for some $\epsilon > 0$. In this case $t + \epsilon$ would be a profitable deviation. Hence, $t = w^1$. Furthermore, equilibrium requires $\sigma(t) = 1$. Notice this offer makes buyers indifferent between accepting or rejecting it. Besides, any other offer on the equilibrium path is rejected with some positive probability. Therefore, buyers get no fraction of the surplus either way. Hence, $V = 0$. The payoff of low-quality sellers is also constant across equilibria: $U^1 = v^1 - c^1$. Consequently, $r^1 = (1 - \delta)c^1 + \delta v^1$.

An equilibrium is separating if $S^1 \cap S^2 = \emptyset$. From Proposition 4, $S^1 = \{v^1\}$ and $S^2 = \{v^2\}$. The offer from high-quality sellers is accepted with probability:

$$\sigma(v^2) \in (0, \frac{v^1 - r^1}{v^2 - r^1}] = (0, \frac{(1 - \delta)(v^1 - c^1)}{v^2 - \delta v^1 - (1 - \delta)c^1}] \tag{5}$$

Just as predicted by Corollary 1, $\sigma(v^2) \to 0$ as $\delta \to 1$. This is not necessarily the case for $U^2$. In a separating equilibrium, the payoff of a high-quality seller is not affected by $\delta$:

$$U^2 \in (0, \frac{v^1 - c^1}{v^2 - c^1}(v^2 - c^2)] \tag{6}$$

If $v^1 = c^1$, then every equilibrium is separating. Otherwise, there are also semi-separating equilibria, in which $S^1 = \{v^1, t^*\}$ and $t^* \in S^2$ for some $t^* \in [c^2, v^2)$. This common offer is accepted with probability:

$$\sigma(t^*) = \frac{v^1 - r^1}{t^* - r^1} = \frac{(1 - \delta)(v^1 - c^1)}{t^* - \delta v^1 - (1 - \delta)c^1} \tag{7}$$

High-quality sellers' payoff is an increasing function of $t^*$. However, it is always below the highest payoff associated with a separating equilibrium. Therefore, when there are only two types, the Riley equilibrium is Pareto efficient in the class of all equilibria. It also achieves the highest equilibrium welfare.

In order to fully appreciate the consequences of signaling through prices in decentralized markets, it is interesting to compare these results with those from Moreno and Wooders (2010), where signaling is not present. If we now assume that, instead of sellers, buyers set prices after agents are matched, then equilibrium prices are either equal to sellers' reservation prices, $r^1$ and $r^2$, or, if buyers get no surplus ($V = 0$), some other price that is necessarily rejected by both types of sellers.[7] Moreno and Wooders (2010) show that, if $\delta$ is sufficiently close to 1, then there is an equilibrium in which buyers offer all three prices with positive probabilities. Furthermore, if $v^2 - c^2 > v^1 - c^1$, then this equilibrium is unique.

In this equilibrium, type 2 sellers never extract a positive surplus. Hence, $U^2 = 0$ and $r^2 = c^2$. Since buyers offer rejected prices with positive probability, $V = 0$. Hence, $\tilde{f}v^2 + (1 - \tilde{f})v^1 - c^2 = 0$, which implies that $\tilde{f} = \hat{f} \equiv \frac{v^1 - c^1}{v^2 - v^1}$. In that way, adverse selection is not severe in the steady-state market composition. Furthermore, $(1 - \tilde{f})(v^1 - r^1) = 0$. Since $r^1 = c^1 + \delta U^1$,

$$W = U^1 = \frac{v^1 - c^1}{\delta} \tag{8}$$

Therefore, as $\delta \to 1$, only gains from trade with the lowest type are realized. This is in striking contrast to our results for decentralized markets with signaling, where limit welfare may be as high as $(1 + \frac{v^2 - c^2}{v^2 - c^1})(v^1 - c^1)$.

# 6 CONCLUSION

In this paper, we have analyzed signaling in decentralized dynamic markets with adverse selection. We have shown that prices signal quality and that welfare is not affected by the level of frictions. Unlike models in which the uninformed party makes all price offers, welfare does not decrease as frictions vanish. Therefore, signaling is relevant to explain prices, trade and payoffs in these markets.

We have considered only an extreme framework in which prices are set by informed sellers. A general model would allow for offers from both sellers and buyers. Such analysis is left for future research.

# REFERENCES


[1] BAGWELL, K.; RIORDAN, M.H. (1991): “High and Decling Prices Signal Product Quality," *American Economic Review*, v. 81, n. 1, p. 224-239.

[2] CAMARGO, B.; LESTER,B. (2013): “Trading Dynamics in Decentralized Markets with Adverse Selection," mimeo.

[3] CHIU, J.; KOEPPL,T. (2011): “Trading Dynamics with Adverse Selection and Search: Market Freeze, Intervention and Recovery," mimeo.

[4] DALEY, B.; GREEN,B. (2012): “Waiting for News in the Market for Lemons," *Econometrica*, v. 80, n.4, p.1433-1504.


[7] If a price $t$ is above $r^2$, then it is accepted by both types of seller. By slightly reducing the offer to $t'$, $r^2 < t' < t$, a buyer would have her offer accepted at the same probability while extracting a large surplus from sellers. Hence, offering $t$ is not optimal. Similarly, offering any price in $(r^1, r^2)$ is not optimal. Hence, prices are either $r^1$, $r^2$ or below $r^1$.

[5] DENECKERE, R.; LIANG,M. (2006): “Bargaining with Interdependent Values," *Econometrica*, v. 74, n. 5, p. 1309-1364.

[6] ELLINGSEN, T. (1997): “Price Signals Quality: The Case of Perfectly Inelastic Demand," *International Journal of Industrial Organization*, v. 16, n.1 , p. 43-61.

[7] FUCHS, W.; SKRZYPACZ, A. (2014): “Government Interventions in a Dynamic Market with Adverse Selection," mimeo.

[8] GERARDI, D.; HORNER, J.; MAESTRI, L. (2013): “The Role of Commitment in Bilateral Trade," mimeo.

[9] GUERRIERI, V.,; SHIMER, R. (2013): “Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality," *American Economic Review*, forthcoming.

[10] HORNER, J.; VIEILLE, N. (2009): “Public vs. Private Offers in the Market for Lemons," *Econometrica*, v. 77, n. 1, p. 26-69.

[11] JANSSEN, M. C. W.; ROY, S. (2002): “Dynamic Trading in a Durable Good Market with Asymmetric Information," *International Economic Review*, v. 43, n. 1, p. 257-282.

[12] KAYA, A.; KIM, K. (2014): “Trading Dynamics in the Market for Lemons," mimeo.

[13] KIM, K. (2012): “Information about Sellers' Past Behavior in the Market for Lemons," mimeo.

[14] KURLAT, P. (2013): “Lemons Market and the Transmission of Aggregate Shocks," *American Economic Review*, v. 103, n. 4, p. 1463-1489.

[15] MORENO, D.; WOODERS, J. (2010): “Decentralized Trade Mitigates the Lemons Problem," *International Economic Review*. v. 51, n. 2, p. 383-399.

# APPENDIX

### PROOF OF LEMMA 1

Item (i) is trivial. We prove item (ii). Let $i < n$ and suppose, by contradiction, that $U^i = 0$. The stationarity condition (vi) implies there is $t \in S^n$ such that $\sigma(t) > 0$. From item (i), $t \geq r^n \geq c^n > c^i = r^i$. By offering $t$, type $i$ sellers would get a positive payoff. Contradiction.

### PROOF OF LEMMA 2

Let $i > j$. We first show that $r^i > r^j$. Suppose, by contradiction, that $r^j \geq r^i$. Hence $U^j - U^i \geq \frac{c^i - c^j}{\delta}$. Consider $t \in S^i$ and $t' \in S^j$. Then $(1-\delta)(U^j - U^i) = \sigma(t')(t' - r^j) - \sigma(t)(t - r^i)$.

Hence:

$$\sigma(t')(t'-r^j)-\sigma(t)(t-r^i)\geq\frac{(1-\delta)}{\delta}(c^i-c^j)$$

This implies $\sigma(t')(t'-r^j)>\sigma(t)(t-r^i)$. However, optimality of type $i$ sellers' offers requires $\sigma(t)(t-r^i)\geq\sigma(t')(t'-r^i)\geq\sigma(t')(t'-r^j)$. Contradiction.

We now prove that $S^i\geq S^j$. Let $t\in S^i$ and $t'\in S^j$. Hence, $\sigma(t)(t-r^i)\geq\sigma(t)(t'-r^i)$ and $\sigma(t)(t-r^j)\leq\sigma(t')(t'-r^j)$. Subtracting the latter inequality from the first, we get $[\sigma(t)-\sigma(t')](r^j-r^i)\geq 0$. By Lemma 1, $\sigma(t')>0$. Since $r^i>r^j$, $\sigma(t')\geq\sigma(t)$. This implies that $t\geq t'$ if $\sigma(t)>0$. From Lemma 1, this is necessarily the case if $i<n$. If $i=n$, then it is possible that $\sigma(t)=0$. In this case, $t\notin S^j$ for any $j<n$. Buyers reject such offer only if $t\geq w^n$. Since $t'$ is accepted with positive probability, $t'\leq w^n$. Hence $t\geq t'$.

We now show that $U^i<U^j$. For any $t\in S^i$ and any $t'\in S^j$:

$$U^j=\frac{\sigma(t')(t'-r^j)}{1-\delta}\geq\frac{\sigma(t)(t-r^j)}{1-\delta}$$

. If $\sigma(t)=0$, then $U^i=0<U^j$, where the inequality follows from Lemma 1. If $\sigma(t)>0$, then $r^i>r^j$ implies that:

$$\frac{\sigma(t)(t-r^j)}{1-\delta}>\frac{\sigma(t)(t-r^i)}{1-\delta}=U^i$$

Hence, $U^j>U^i$.

## LEMMA 5 AND PROOF

**Lemma 5.** *$S^i$ has at most three elements if $1<i<n$ and at most two elements if $i=1$. Furthermore, $S^n$ has at most two elements that are accepted with positive probability.*

*Proof.* Consider the lowest type 1. A direct consequence of Lemma 2 is that only the highest offer in $S^1$ may intersect any other $S^i$. Suppose there are other two offers in $S^1$, $t$ and $t'$, that are below the highest offer. Hence $\pi^1(t)=\pi^1(t')=1$. Since, from Lemma 1, $\sigma(t)>0$ and $\sigma(t')>0$, indifference implies $t=t'$. An argument very similar establishes that $S^i$ has at most three elements for $1<i<n$. Consider now $S^n$. Only its lowest offer may intersect any other $S^i$. Consider there is other offer $t$ in $S^n$ that satisfies $t\leq w^n$. Hence, $\pi^n(t)=1$. If $t<w^n$, then $\sigma(t)=1$. In this way $n$ would have an incentive not to offer the lowest offer, which is a contradiction. Hence $t=w^n$. This implies the desired result. □

## PROOF OF LEMMA 3

Fix $t^1<t^2<...<t^m$ and $I_1\leq I_2\leq...\leq I_m$. Suppose that $\sigma(t_k)(t_k-r^{\bar{i}(k)})\geq\sigma(t_{k+1})(t_{k+1}-r^{\bar{i}(k)})$ for every $k<m$. Hence, a seller of type $\bar{i}(k)$ has no incentive to offer $t_{k+1}$. We show that, for any $h>k$, she has no incentive to offer $t_h$ as well. The proof is by induction. We have already proved the desired result for $h=k+1$. Suppose it holds for a given $k>l$. Then:

$$\sigma(t_h) \leq \frac{t_l - r^{\bar{i}(k)}}{t_h - r^{\bar{i}(k)}}\sigma(t_k)$$

We now show it holds for $h+1$:

$$\sigma(t_{h+1}) \leq \frac{t_h - r^{\bar{i}(h)}}{t_{h+1} - r^{\bar{i}(h)}} \cdot \frac{t_k - r^{\bar{i}(k)}}{t_h - r^{\bar{i}(k)}}\sigma(t_k)$$

$$= \frac{t_h - r^{\bar{i}(h)}}{t_{h+1} - r^{\bar{i}(h)}} \cdot \frac{t_{h+1} - r^{\bar{i}(k)}}{t_h - r^{\bar{i}(k)}} \cdot \frac{t_l - r^{\bar{i}(k)}}{t_{h+1} - r^{\bar{i}(k)}}\sigma(t_k) < \frac{t_l - r^{\bar{i}(k)}}{t_{h+1} - r^{\bar{i}(k)}}\sigma(t_k)$$

Hence, $\sigma(t_k)(t_k - r^{\bar{i}(k)}) \geq \sigma(t_h)(t_h - r^{\bar{i}(k)})$ for every $1 \leq k < h \leq m$. Consider $i \in I_k$. From Lemma 2, $r^i \leq \bar{i}(k)$, which implies that $\frac{t_k - r^i}{t_h - r^i} \leq \frac{t_k - r^{\bar{i}(k)}}{t_h - r^{\bar{i}(k)}}$. Hence, $\sigma(t_k)(t_k - r^i) \geq \sigma(t_h)(t_h - r^i)$ for every $i \in I_k$ and for every $1 \leq k < h \leq m$.

Now suppose that $\sigma(t_{k+1})(t_{k+1} - r^{\underline{i}(k+1)}) \geq \sigma(t_k)(t_k - r^{\underline{i}(k+1)})$ for every $k < m$. In a symmetric way one may show that $\sigma(t_k)(t_k - r^i) \geq \sigma(t_h)(t_h - r^i)$ for every $i \in I_k$ and for every $1 \leq h < k \leq m$.

Hence,

$$\sigma(t_k)(t_k - r^i) \geq \sigma(t_h)(t_h - r^i) \qquad \forall\, i \in I_k\,, \forall\, k, h \in \{1, ..., m\}$$

## LEMMA 6 AND PROOF

Although the following result is too technical to be included in the main text, it is very important to prove Propositions 1 and 2.

**Lemma 6.** *Suppose* $S^k = \{t\}$ *and let* $j = max\,\{i \in I : t \in S^i\}$. *Then* $\{f^i_{(k,j)}\}_{i=1}^n$ *first-order stochastically dominates* $\{\pi^i(t)\}_{i=1}^n$.

*Proof.* Consider $S^k = \{t\}$ and let $j = max\,\{i : t \in S^i\}$. Pooling from any type below $k$ reduces $\{\pi^i(t)\}$. Therefore, it is enough to prove the result for the case in which $t \notin S^i$ for any $i < j$. The case $j = k$ is trivial. Hence, suppose $j > k$. From Lemma 2, $S^i = \{t\}$ for every $k < i < j$. Let $\alpha$ be the probability type $j$ offers $t$. With probability $1 - \alpha$, she offers something else that is accepted with average probability $\sigma^j < \sigma(t)$. Define $\theta \equiv \frac{\alpha}{\alpha + (1-\alpha)\frac{\sigma^j}{\sigma(t)}} < 1$. Stationarity implies that:

$$N^i = \frac{f^i}{\sigma(t)} i < j$$

$$\alpha N^j = \theta \frac{f^j}{\sigma(t)}$$

We characterize beliefs at $t$ by Bayes' rule:

$$\pi^i(t) = \begin{cases} \frac{N^i}{\sum_{l=k}^{j-1} N^l + \alpha N^j}\text{ , if } k \leq i < j \\ \frac{\alpha N^i}{\sum_{l=1}^{j-1} N^l + \alpha N^j}\text{ , if } i = j \\ 0\text{ , if } i > j \text{ or } i < k \end{cases}$$

Hence, $\pi^i(t) = \frac{f^i}{\sum_{l=k}^{j-1} f^l + f^j \theta} > f^i_{(k,j)}$ for every $k \leq i < j$. This proves that $\{f^i_{(k,j)}\}_{i=1}^n$ first-order stochastically dominates $\{\pi^i(t)\}_{i=1}^n$. □

## PROOF OF PROPOSITION 1

We prove necessity first. Suppose (1) is violated. Then there is some $k \in I_0 \setminus \{n\}$ such that $\sum_{i=1}^n f^i_{(k,j)} v^i < c^j$ for every $j > k$. Suppose there is an equilibrium. Since $k < n$, $U^k > 0$ by Lemma 1. This implies that any offer $t \in S^k$ is strictly greater than $c^k \geq v^k$. Furthermore, $\sigma(t) > 0$. Hence:

$$\sum_{i=1}^n \pi^i(t) w^i \geq t > c^k \geq v^k$$

Hence there is $i > k$ such that $t \in S^j$. Let $j$ be the greatest of these types. From Lemma 2, $S^i = \{t\}$ for any type $k < i < j$. Notice that $S^k = \{t\}$ as well, since from Lemma 2 there is a single offer in $S^k$ that pools with higher types. Applying Lemma 6,

$$\sum_{i=1}^n \pi^i(t) v^i \leq \sum_{i=1}^n f^i_{(k,j)} v^i < c^j \leq r^j$$

From Lemma 1, $\sigma(t) = 0$. Contradiction. Hence, there is no equilibrium.

Now suppose (2) is violated ($v^n < c^n$) and that there is an equilibrium. By stationarity, there is an offer $t \in S^n$ such that $\sigma(t) > 0$. Hence, from Lemma 1, $t \geq r^n \geq c^n > v^n \geq w^n$. Since $\sum_{i=1}^n \pi^i(t) w^i \leq w^n < t$, then $\sigma(t) = 0$. Contradiction.

We now prove sufficiency. The proof is constructive. As noticed in Section 3, we do not need to concern ourselves with off the equilibrium path deviations. Let $\hat{I}_0 \equiv I_0 \setminus \{n\}$. By assumption, for each $k \in \hat{I}_0$ there is $j^k > k$ such that $\sum_{i=1}^n f^i_{(k,j^k)} v^i \geq c^{j^k}$. Let $J$ be the set of all such $j^k$. Notice $J \cap \hat{I}_0 = \emptyset$.

In order to construct an equilibrium, we separate types in groups, such that types that belong to a same group make the same offer. Let $\phi : \hat{I}_0 \to J$ be the function that associates $k \in \hat{I}_0$ to the lowest $j \in J$ that satisfies $j > k$, i.e, $\phi(k) = min\{j \in J : j > k\}$.

For each $j \in \phi(\hat{I}_0)$, let $A_j \equiv \{i \in I : min\, \phi^{-1}(j) \leq i \leq j\}$. In the proposed equilibrium, every type in $A_j$ offers (with certainty) $\sum_{i=1}^n f^i_{(min\ \phi^{-1}(j),j)} v^i$. Any other type $i \in I \setminus (\cup_{j \in \phi(\hat{I}_0)} A^j)$ offer $v^i$.

It is staightforward to see that buyers are indifferent between accepting or rejecting any offer that is made with positive probability. Furthermore, there is a finite number of offers that may be listed in an increasing order: $\{t_1, ..., t_m\}$. Let $I_h$ be the set of types that make an offer $t_h$. Hence $\{I_1, ..., I_m\}$ is also in an increasing order. For each $k$, let $\bar{i}(h) \equiv maxI_h$ and $\underline{i}(h) \equiv minI_h$.

It remains to adjust trade probabilities $\{\sigma(t_h)\}_{h=1}^m$ such that no type is willing to deviate. Let $\sigma(t_1) = 1$ and define recursively:

$$\sigma(t_{h+1}) = \frac{t_h - r^{\bar{i}(h)}}{t_{h+1} - r^{\bar{i}(h)}}\sigma(t_h) \quad \forall h < m$$

From Lemma 3, no type has an incentive to deviate. Therefore, we have constructed an equilibrium.

**PROOF OF PROPOSITION 2**

The proof is by contradiction.

Suppose there is no $t \in S^1$ such that $t < S^{i^*}$. Then $S^1 = \{t\}$ and $t \in S^{i^*}$, from Lemma 2. Stationarity requires that $\sigma(t) > 0$. Let $j$ be the greatest type such that $t \in S^j$. Necessarily, $j \geq i^*$. From Lemma 6,

$$\sum_{i=1}^{n} \pi^i(t)v^i \leq \sum_{i=1}^{n} f^i_{(1,j)}v^i < c^j \leq r^j \leq t$$

where the last inequality follows from Lemma 1. However, the expression above implies that buyers reject $t$, i.e., that $\sigma(t) = 0$. Contradiction.

Hence there is $t \in S^1$ such that $t < S^{i^*}$. From Lemma 2, $t < S^j$ for every $j \geq i^*$.

**PROOF OF COROLLARY 1**

Let $\delta_k \to 1$ and fix a sequence of equilibria. From Proposition 2, for all $k \in \mathbb{N}$ there is $t_k \in S^1_k$ such that $t_k < S^{i^*}_k$. Consider $t^{i^*}_k \in S^{i^*}_k$. Hence $t^{i^*}_k > t_k$.

From Lemmas 2 and 6, the highest value that $t_k$ takes in equilibrium is:

$$t_k \leq \sum_{i=1}^{n} f^i_{(1,i^*-1)}v^i \equiv \bar{t} \leq \sum_{i=1}^{n} f^i_{(1,i^*)}v^i < c^{i^*}$$

Since type 1 sellers offer $t_k$, $\sigma_k(t^{i^*}_k) \leq \frac{t_k - r^1_k}{t^{i^*}_k - r^1_k}\sigma_k(t_k)$, that is equivalent to:

$$\kappa_k(t^{i^*}_k) \equiv \frac{\sigma_k(t^{i^*}_k)}{1 - \delta_k(1 - \sigma_k(t^{i^*}_k)} \leq \frac{t_k - c^1}{t^{i^*}_k - c^1}\kappa_k(t_k) \leq \frac{\bar{t} - c^i}{c^{i^*} - c^i} < 1$$

Hence, since $\kappa_k(t^{i^*}_k)$ is bounded by some value below 1, then $\sigma_k(t^{i^*}_k) \to 0$ as $\delta_k \to 1$.

**LEMMA 7 AND PROOF**

For every offer $t$, let $\kappa(t) \equiv \frac{\sigma(t)}{1-\delta(1-\sigma(t))}$. The next result is essential to characterize equilibrium welfare:

**Lemma 7.** *$W \leq \sum_{i=1}^{n} f^i \int_T \kappa(t)(v^i - c^i)dG^i(t)$, with equality if $V = 0$.*

*Proof.* The result is trivial for $V = 0$. Consider, then, an equilibrium with $V > 0$. We may represent it with an equilibrium in the type space $(J, (\varphi^j)_{j\in J})$, in which sellers' strategies are pure. From Lemma 5, there is a finite set of offers $\{t_1, ..., t_m\}$ that are accepted with positive probabilities. Let $J_k$ be the set of types that offer $t_k$. Notice that $\sum_{j\in\cup_{k=1}^{m} J_k} \varphi^j = 1$. Only the lowest offer, $t_1$, is accepted with probability 1. Hence, $\kappa(t^1) = 1$. By definition, buyers' payoff is:

$$V = \frac{\sum_{j\in J_1} N^j(w^j - t_1)}{\sum_{i\in J} N^i} + \delta V$$

Since $w^j = v^j - \delta V$ for every $j \in J$ and, from stationarity, $N^j = \varphi^j$ for every $j \in J_1$, we may rewrite buyers' payoff as:

$$V = \frac{\sum_{j\in J_1} \varphi^j(v^j - t_1)}{\sum_{j\in J_1} \varphi^j + (1-\delta)\sum_{i\in J\backslash J_1} N^i}$$

We claim that $V \leq \sum_{j\in J_1} \varphi^j(v^j - t_1)$. Notice that:

$$\sum_{j\in J_1} \varphi^j + (1-\delta)\sum_{i\in J\backslash J_1} N^i = \sum_{j\in J_1} \varphi^j + (1-\delta)(\sum_{j\in J_2} \frac{\varphi^j}{\sigma(t_2)} + ... + \sum_{j\in J_m} \frac{\varphi^j}{\sigma(t_m)})$$

$$\geq \sum_{j\in J_1} \varphi^j + (1-\delta)\sum_{j\in J\backslash J_1} \frac{\varphi^j}{\sigma(t_2)}$$

Let $j^* \equiv max J_1$. Then $r^{j^*} = (1-\delta)c^{j^*} + \delta t_1$. Since offering $t_2$ is not a profitable deviation for $j^*$,

$$\sigma(t_2) \leq \frac{t_1 - r^{j^*}}{t_2 - r^{j^*}} = \frac{(1-\delta)(t_1 - c^{j^*})}{t_2 - \delta t_1 - (1-\delta)c^{j^*}} \leq 1 - \delta$$

Hence, $\sum_{j\in J_1} \varphi^j + (1-\delta)\sum_{i\in J\backslash J_1} N^i \leq \sum_{j\in J} \varphi^j = 1$. This proves the claim. Hence, $V + \sum_{j\in J_1} \varphi^j U^j \leq \sum_{j\in J_1} \varphi^j(v^j - c^j)$. Besides, for $k > 1$, since $w^j < v^j$, $\sum_{j\in J_k} \kappa(t_k)\varphi^j(v^j - c^j) > \sum_{j\in J_k} \varphi^j U^j$. Summing up all these inequalities, one gets the desired result. □

**PROOF OF LEMMA 4**

Fix $\delta \in [0, 1)$ and consider an equilibrium that satisfies $V = 0$. Consider the type space $(J, (\varphi^j)_{j\in J})$, such that sellers' strategies are pure. In this case, denote by $t^j$ the single offer made by type $j$ sellers.

Consider $\delta' \in [0, 1)$. We construct an equilibrium with the same payoffs. Suppose sellers' strategies are the same. Hence, beliefs on the equilibrium path are also the same and buyers are indifferent between accepting or rejecting any of these offers. Hence, $V' = 0$.

We now define trade probabilities: for every $t \in \cup_{j\in J}\{t^j\}$, let:

$$\frac{\sigma'(t)}{1-\delta'(1-\sigma'(t))} = \frac{\sigma(t)}{1-\delta(1-\sigma(t))} \equiv \kappa(t)$$

Hence, $U^{j'} = \kappa(t^j)(t^j - c^j) = U^j$ for every $j \in J$. It remains to show that there are no profitable deviations. We are concerned only with deviations on the equilibrium path. For every $i, j \in J$, $\sigma(t^i)(t^i - r^i) \geq \sigma(t^j)(t^j - r^i)$, that is equivalent to $U^i - U^j \geq \kappa(t^j)(c^j - c^i)$. Since this last inequality is independent of $\delta$, it is also equivalent to $\sigma'(t^i)(t^i - r^{i'}) \geq \sigma'(t^j)(t^j - r^{i'})$.

## PROOF OF PROPOSITION 3

We first prove the following lemmas:

**Lemma 8.** *For every equilibrium with payoffs $(V, U^1, ..., U^n)$, there is another equilibrium with payoffs that satisfy $\hat{V} = 0$, $\hat{W} \geq W$ and $\hat{U}^i \geq U^i$ for every $i \in I$.*

*Proof.* Consider the equilibrium with pure strategies for sellers in $(J, (\varphi^j)_{j\in J})$. In this case, we may partition $J$ in $J_1, ..., J_m$ such that all types $j \in J_k$ offer the same price $t_k$. Let $\bar{j}(k) = max\ J_l$ and $\underline{j}(k) = min\ J_l$.

From stationarity, $\sigma(t_k) > 0$ for every $k \in \{1, ..., m\}$. From Lemma 3, only local deviations are relevant:

$$\frac{t_k - r^{\underline{j}(k+1)}}{t_{k+1} - r^{\underline{j}(k+1)}}\sigma(t_k) \leq \sigma(t_{k+1}) \leq \frac{t_k - r^{\bar{j}(k)}}{t_{k+1} - r^{\bar{j}(k)}}\sigma(t_k) \quad \forall\, l < m$$

Defining $\kappa(t) \equiv \frac{\sigma(t)}{1-\delta(1-\sigma(t))}$, then the condition above is equivalent to:

$$\frac{t_k - c^{\underline{j}(k+1)}}{t_{k+1} - c^{\underline{j}(k+1)}}\kappa(t_k) \leq \kappa(t_{k+1}) \leq \frac{t_k - c^{\bar{j}(k)}}{t_{k+1} - c^{\bar{j}(k)}}\kappa(t_k) \quad \forall\, l < m$$

Since $V \geq 0$, $t_1 \leq \sum_{j\in J} \pi^j(t_1)w^j = \sum_{j\in J} \varphi^j_{(1,\bar{j}(1))} w^j$. For $k > 1$, since $\sigma(t_k) < 1$, $t_k \leq \sum_{j\in J} \pi^j(t_k)w^j = \sum_{j\in J} \varphi^j_{(\underline{j}(k),\bar{j}(k))} w^j$.

We propose the following equilibrium: if $j \in J_k$, then $\hat{S}^j = \{\hat{t}_k\}$, where $\hat{t}_k \equiv \sum_{j\in J} \varphi^j_{(\underline{j}(k),\bar{j}(k))} v^j$. Notice that $\hat{t}_1 \geq t_1 + \delta V$ and $\hat{t}_k = t_k + \delta V$ for $k > 1$.

Let $\hat{\kappa}(\hat{t}_1) = 1$ and, for every $k < m$, let

$$\hat{\kappa}(\hat{t}_{k+1}) = \frac{\hat{t}_k - c^{\bar{j}(k)}}{\hat{t}_{k+1} - c^{\bar{j}(k)}}\hat{\kappa}(\hat{t}_k)$$

We show by induction that $\hat{\kappa}(\hat{t}_k) \geq \kappa(t_k)$ for every $k \in \{1, ..., m\}$. Clearly this inequality holds for $k = 1$. Suppose it holds for some $k$. Hence:

$$\hat{\kappa}(\hat{t}_{k+1}) = \frac{\hat{t}_k - c^{\bar{j}(k)}}{\hat{t}_{k+1} - c^{\bar{j}(k)}}\hat{\kappa}(\hat{t}_k) \geq \frac{t_k + \delta V - c^{\bar{j}(k)}}{t_{k+1} + \delta V - c^{\bar{j}(k)}}\kappa(t_k) \geq \frac{t_k - c^{\bar{j}(k)}}{t_{k+1} - c^{\bar{j}(k)}}\kappa(t_k) \geq \kappa(t_{k+1})$$

From Lemma 7, $\hat{W} \geq W$, since $\hat{V} = 0$. Furthermore, since $\hat{t}_k \geq t_k$ and $\hat{\kappa}(\hat{t}_k) \geq \kappa(t_k)$ for every $k \in \{1, ..., m\}$, then $\hat{U}^j \geq U^j$ for every $j \in J$. □

**Lemma 9.** *For every $(V, U^1, ..., U^n) \in \mathbb{U}_0$, there is an equilibrium in pure strategies for sellers and $\hat{V} = 0$ that satisfies $\hat{U}^i \geq U^i$ for every $i \in I$.*

*Proof.* Fix $\delta \in [0, 1)$ and consider an equilibrium with payoffs $(V, U^1, ..., U^n) \in \mathbb{U}_0$. From Lemma 5, each $S^i$ has a lowest element, which we denote as $\underline{t}^i$. We may partition $I$ in $I_1, ..., I_m$ (in increasing order) such that $i, j \in I_k$ if $\underline{t}^i = \underline{t}^j$. We denote this common lowest offer by $\underline{t}_k$. Let the highest type in $I_k$ be $\bar{i}(k)$ and the lowest one $\underline{i}(k)$.

We propose an equilibrium in pure strategies for sellers and $\hat{V} = 0$ that Pareto dominates the first equilibrium. We first define sellers' strategies: if $i \in I_k$, then $\hat{S}^i = \{t_k\}$, where $t_k \equiv \sum_{i=1}^n f^i_{(\underline{i}(k), \bar{i}(k))} v^i$. Notice that buyers are always indifferent between accepting or rejeting any of these offers. Hence, $\hat{V} = 0$. Besides, from Lemma 6, $t_k \geq \underline{t}_k$ for every $k \in \{1, ..., m\}$.

Now we propose the following probabilities of trade:

$$\hat{\sigma}(t_k) = \begin{cases} 1, \text{ if } k = 1 \\ \frac{t_{k-1} - \hat{r}^{\bar{i}(k-1)}}{t_k - \hat{r}^{\bar{i}(k-1)}}\hat{\sigma}(t_k), \text{ if } l > 1 \end{cases}$$

where $\hat{r}^i = c^i + \delta\hat{U}^i$. From Lemma 3, no type of seller is willing to deviate to an offer on the equilibrium path. Hence, this is an equilibrium.

It remains to show that $\hat{U}^i \geq U^i$ for every $i \in I$. The proof is by induction. For $i \in I_1$, $1 = \hat{\sigma}(t_1) \geq \sigma(\underline{t}_1)$. Since $t_1 \geq \underline{t}_1$, then $\hat{U}^i \geq U^i$.

Now suppose that $\hat{U}^i \geq U^u$ for every $i \in I_k$. In particular, $\hat{U}^{\bar{i}(k)} \geq U^{\bar{i}(k)}$. Since $\underline{t}_k \in S^{\bar{i}(k)}$, then $\sigma(\underline{t}_k)(\underline{t}_k - r^{\bar{i}(k)}) \geq \sigma(\underline{t}_{k+1})(\underline{t}_{k+1} - r^{\bar{i}(k)})$, that is equivalent to $U^{\bar{i}(k)} \geq \kappa(\underline{t}_{k+1})(\underline{t}_{k+1} - c^{\bar{i}(k)})$. Since $U^i = \kappa(\underline{t}_{k+1})(\underline{t}_{k+1} - c^i)$ for every $i \in I_{k+1}$:

$$U^i \leq U^{\bar{i}(k)}\frac{\underline{t}_{k+1} - c^i}{\underline{t}_{k+1} - c^{\bar{i}(k)}} \leq \hat{U}^{\bar{i}(k)}\frac{\underline{t}_{k+1} - c^i}{\underline{t}_{k+1} - c^{\bar{i}(k)}}$$

$$\leq \hat{U}^{\bar{i}(k)}\frac{t_{k+1} - c^i}{t_k - c^{\bar{i}(k)}} = \hat{\kappa}(t_{k+1})(t_{k+1} - c^i) = \hat{U}^i$$

This proves the desired result. □

Now we prove Proposition 3. From Lemmas 4 and 8, $\overline{W}(\delta) = sup\ \mathbb{W}(\delta)$ is invariant to $\delta$, since $sup\ \mathbb{W}(\delta) = sup\ \mathbb{W}_0$, where $\mathbb{W}_0$ is the set of welfare achieved by equilibria with $V = 0$. Denote by $\overline{W}$ this supremum. We show there is an equilibrium that achieves welfare $\overline{W}$. From Lemma 9, we have to consider only equilibria in which sellers play pure strategies.

Notice that there is a finite set of sellers' pure strategies that are played in some equilibrium with $V = 0$. We denote this set by $\Gamma$, with typical element $\gamma \equiv (t^1, ..., t^n)$. For each $\gamma \in \Gamma$, define:

$$D^\gamma \equiv \{\kappa \in [0,1]^n : \kappa^i(t^i - c^i) \geq v^1 - c^1 \,\forall\, i \in I,\ \kappa^i(t^i - c^i) \geq \kappa^j(t^j - c^i) \,\forall\, i,j \in I\}$$

It is straightforward to show that $D^\gamma$ is compact. Furthermore, every $(\kappa^1, ..., \kappa^n) \in D^\gamma$ that satisfies $\kappa^i > 0$ for every $i \in I$ is associated to some equilibrium in which sellers play $\gamma$. Let $D \equiv \cup_{\gamma \in \Gamma} D^\gamma$. Since each $D^\gamma$ is compact, then so is $D$. Consider the function $\psi : [0,1]^n \to \Re$ such that $\psi(\kappa^1, ..., \kappa^n) \sum_{i=1}^n f^i \kappa^i (v^i - c^i)$. From Lemma 7, this function associates every equilibrium with its welfare. Since $\psi$ is continuous, $\psi(D)$ is compact. Let $\overline{W} \equiv max\ \psi(D)$. Hence, there is $\kappa \in D$ such that $\psi(\kappa) = \overline{W}$. It is easy to show that $\kappa^i > 0$ for every $i \in I$. Hence, there is an equilibrium that achieves welfare $\overline{W}$.

## PROOF OF PROPOSITION 4

We first characterize quasi-separating equilibria. Consider a quasi-separating equilibrium. In this case, if $v^i > v^1$, $i < n$ and $t \in S^i$, then $0 < \sigma(t) < 1$. For this offer, buyers are indifferent between accepting or rejecting it: hence $\sum_{l=1}^n \pi^l(t) w^l = w^i = t$. Hence, for any $i < n$ such that $v^i > v^1$, $S^i = \{w^i\}$. Since type $n$ sellers also make offers accepted with positive probability, $w^n \in S^n$. Consider now an offer $t$ made by type 1 sellers. Since it is accepted with positive probability, $t \leq w^1$. If the inequality is strict, there would be $\epsilon > 0$ such that $\sigma(t+\epsilon) = 1$, which would be a profitable deviation. Hence, $S^1 = \{w^1\}$. Furthermore, $\sigma(w^1) = 1$, since otherwise type 1 sellers could make an offer slightly smaller that would be accepted with probability one. Hence, $V = 0$ and $S^i = \{v^i\}$ for every $i \in I$. From stationarity, $\sigma(v^i) > 0$ for every $i \in I$.

We know that only deviations on the equilibrium path are relevant. From Lemma 3, these deviations are satisfied if, for every $i > 1$:

$$\frac{v^{i-1} - r^i}{v^i - r^i}\sigma(v^{i-1}) \leq \sigma(v^i) \leq \frac{v^{i-1} - r^{i-1}}{v^i - r^{i-1}}\sigma(v^{i-1})$$

We now use the characterization of a quasi-separating equilibrium in order to prove its existence. Sufficiency is straightforward: any list of strategies, beliefs and value functions that satisfies the conditions above is an equilibrium. It remains to show necessity.

Assume that $v^n < c^n$ and suppose, by contradiction, that there is a quasi-separating equilibrium. In this case, $S^n = \{v^n\}$. Since $r^n \geq c^n > v^n$, then $\sigma(v^n) = 0$. This contradicts stationarity.

Now let $I_0 \setminus \{n\} \neq \emptyset$ and suppose there is a quasi-separating equilibrium. Let $j \in I_0$, $j < n$. Hence $S^j = \{v^j\}$. Since $v^j \leq c^j \leq r^j$, $U^j \leq 0$. However, it contradicts Lemma 1.