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Endogenous debt constraints in collateralized economies with default penalties

V. Filipe Martins-da-Rocha* Yiannis Vailakis†

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Abstract

In infinite horizon financial markets economies, competitive equilibria fail to exist if one does not impose restrictions on agents' trades that rule out Ponzi schemes. When there is limited commitment and collateral repossession is the unique default punishment, Araujo, Páscoa and Torres-Martínez (2002) proved that Ponzi schemes are ruled out without imposing any exogenous/endogenous debt constraints on agents' trades. Recently Páscoa and Seghir (2009) have shown that this positive result is not robust to the presence of additional default punishments. They provide several examples showing that, in the absence of debt constraints, harsh default penalties may induce agents to run Ponzi schemes that jeopardize equilibrium existence.

The objective of this paper is to close a theoretical gap in the literature by identifying endogenous borrowing constraints that rule out Ponzi schemes and ensure existence of equilibria in a model with limited commitment and (possible) default. We appropriately modify the definition of finitely effective debt constraints, introduced by Levine and Zame (1996) (see also Levine and Zame (2002)), to encompass models with limited commitment, default penalties and collateral. Along this line, we introduce in the setting of Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of actions with *finite equivalent payoffs*. We show that, independently of the level of default penalties, restricting plans to have finite equivalent payoffs rules out Ponzi schemes and guarantees the existence of an equilibrium that is compatible with the minimal ability to borrow and lend that we expect in our model.

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An interesting feature of our debt constraints is that they give rise to budget sets that coincide with the standard budget sets of economies having a collateral structure but no penalties (as defined in Araujo, Páscoa and Torres-Martínez (2002)). This illustrates the hidden relation between finitely effective debt constraints and collateral requirements.

JEL Classification: D52, D91

Keywords: Infinite horizon economies; Incomplete markets; Limited commitment; Default; Debt constraints; Collateral; Ponzi schemes

1 Introduction

One of the main difficulties of extending financial markets economies to an infinite horizon is related to the existence of the so-called Ponzi schemes. In the absence of a terminal date agents would attempt to finance unbounded levels of consumption by renewing their credit at infinite. If such schemes are permitted, the agent's decision problem has no solution. Therefore, without debt constraints that limit the rate at which agents accumulate debt, equilibria fail to exist.

Broadly speaking three approaches have been proposed in the literature to deal with the specification of debt constraints in infinite horizon sequential markets models. The main difference among these lines of research hinges on the specific assumptions made about the enforcement of payments as well as the proposed default punishment.

The first approach, due to Magill and Quinzii (1994), Hernández and Santos (1996) and Levine and Zame (1996) (see also Levine and Zame (2002)), introduces debt constraints in economies where payments are fully enforced and therefore there is no default (even on out of equilibrium paths). Magill and Quinzii (1994) argue in favor of implicit debt constraints that restrict budget sets to include portfolios whose value is a bounded sequence along the event tree. An interesting property of equilibria with implicit debt constraints is that it is always possible to find uniform bounds on short-sales which are non-binding at those equilibria. Moreover, under reasonable assumptions on preferences, equilibria with implicit debt constraints coincide with equilibria with transversality type conditions that are often imposed in macroeconomic models (see Blanchard and Fisher (1989) and Ljungqvist and Sargent (2000)). Hernández and Santos (1996) argue in favor of debt constraints that impose a kind of solvency requirement. Households are allowed to borrow against their current value of future endowment streams. When markets are incomplete,

traders may not agree on current value prices. Hernández and Santos (1996) propose a special way of computing current value prices that takes into account the whole set of non-arbitrage price systems. Levine and Zame (1996) (see also Levine and Zame (2002)) offer an alternative formulation of the solvency requirement. They formalize borrowing constraints that restrict agents' debt to be repayable in finite time, that is, they impose debt constraints that are *finitely effective*. Stated differently, agents' actions are finitely effective when they are budget compatible with the threat that, at any period, agents may be restricted to have access to borrowing only for a finite number of periods. Finitely effective debt constraints provide a general characterization of debt constraints that are compatible with equilibrium. More precisely, Levine and Zame (1996) have shown that any loose and consistent debt constraints (see Levine and Zame (1996) for a precise definition) that rule out Ponzi schemes and ensure existence of an equilibrium reduce to be finitely effective.¹

The second approach, due to Kehoe and Levine (1993) (see also Kehoe and Levine (2001)), Zhang (1997) and Alvarez and Jermann (2000), explores debt constraints in economies where commitment is limited and there is a severe punishment for default: if agents do not honor their debts, they are excluded from participating in the asset markets in future periods. In such a setting the authors argue for self-enforcing constraints (so-called participation constraints) that are tight enough to prevent default at equilibrium but simultaneously are loose enough to allow for as much risk sharing as possible.

The third and most recent approach to deal with Ponzi schemes also considers models with limited commitment. However, contrary to self-enforcing borrowing constraints (à la Alvarez and Jermann (2000)) that prevent default at equilibrium, this research line addresses the issue of Ponzi schemes in economies where default may be consistent with equilibrium. It is motivated by the empirical observation that modern economies experience a substantial amount of default and bankruptcy.² One of the most important and

¹See also Hernández and Santos (1996) for a similar discussion.

²Nowadays, there is a vast literature on default that dates back to the seminal contributions of Shubik (1972), Shubik and Wilson (1977) and Dubey and Shubik (1979). Default was introduced in a general equilibrium setting by Dubey, Geanakoplos and Shubik (1990) and Zame (1993). Modern theoretical contributions on default include among others, Dubey, Geanakoplos and Zame (1995), Geanakoplos (1997), Geanakoplos and Zame (2002), Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003), Dubey, Geanakoplos and Shubik (2005), Fostel and Geanakoplos (2008), Páscoa and Seghir (2009), Ferreira and Torres-Martínez (2010). There are also important contributions on default, collateral and credit constraints in macroeconomics (see Bernanke, Gertler and Gilchrist (1996), Kiyotaki and Moore (1997) and Caballero and Krishnamurthy (2001)). This literature emphasizes the feedback from the fall in collateral prices to a fall in borrowing capacity. Recently, Chatterjee, Corbae, Nakajima and Ros-Rull (2007) and Livshits, MacGee and

widespread means of securing loans and lowering the level of default in financial markets is collateral.³ Araujo, Páscoa and Torres-Martínez (2002) (see also Kubler and Schmedders (2003)) showed that, without imposing any debt constraints or transversality conditions, Ponzi schemes are ruled out in economies where collateral is the only mechanism that enforces agents to (partially) pay their debts. The intuition behind their result is as follows. Combining short-sales with the purchase of collateral constitutes a joint operation that yields non-negative returns.⁴ By non-arbitrage, at equilibrium, the price of the collateral exceeds the price of the asset, implying that collateral costs exceed the value of loans. Therefore, it becomes impossible to pay a previous debt by issuing new debt.

In most economic systems collateral is not the only mean of securing loans. The default option usually entails additional economic consequences.⁵ A possible reason is that the effectiveness of collateral is rather limited in the presence of large negative shocks in the value of collateral guarantees. One approach to model additional enforcement mechanisms is to introduce linear utility penalties (see Dubey, Geanakoplos and Shubik (1990), Zame (1993), Dubey, Geanakoplos and Shubik (2005) and the literature cited therein). These penalties might be interpreted as the consequences (directly assessed in terms of utility) of some third party punishment such as prison terms and pangs of conscience, and/or of some non-modeled economic punishment such as exclusion from credit markets and garnishing of future income.

A surprising result found by Páscoa and Seghir (2009) is that the introduction of default penalties in the model of Araujo, Páscoa and Torres-Martínez (2002) may induce payments besides the value of the collateral and lead to the reappearance of Ponzi schemes. The intuition is simple: when penalties are severe, agents have incentives to repay more than the value of the depreciated

Tertilt (2007) have calibrated macroeconomic models with incomplete markets and default and used them to address various policy issues.

³Collateral-using activities have expanded rapidly in recent years. Financial institutions extensively employ collateral in lending, in securities trading and derivative markets and in payment and settlement systems. Central banks generally require collateral in their credit operations. Common examples of collateralized lending are home mortgages, margin purchases of securities, overnight repurchase agreements and pawn shop loans.

⁴Since there is no other punishment than the seizure of collateral, borrowers will always deliver the minimum between their promises and the value of the associated collateral requirements.

⁵For instance, if an agent files for bankruptcy under Chapter 7 of the U.S. bankruptcy code, the following things may happen (see Chatterjee, Corbae, Nakajima and Ros-Rull (2007)): (1) he is not allowed to save and his existing savings will be completely garnished; (2) he has to pay a proportion of the current income as cost of filing for bankruptcy; (3) a proportion of his current labor income is garnished; (4) his credit history turns bad and he is excluded from the loan market.

collateral. In this case, the joint operation of combining short-sales with the purchase of collateral no longer yields non-negative returns. Therefore, loans may exceed collateral costs and agents may run Ponzi schemes.

One may think that the reappearance of Ponzi schemes is related to the particular additional enforcement mechanism (linear utility penalties) Páscoa and Seghir (2009) have considered. However, Ferreira and Torres-Martínez (2010) showed that any effective additional enforcement mechanism implies the non-existence of physically feasible optimal plans.⁶ That is, any effective additional enforcement mechanism gives rise to Ponzi schemes in infinite horizon collateralized economies. Hence, it is the effectiveness of the mechanism that induces agents to run a Ponzi scheme, not the mechanism per se.

Given the findings of Páscoa and Seghir (2009) and Ferreira and Torres-Martínez (2010) we propose to address the following question: what kind of borrowing constraints rule out Ponzi schemes and ensure existence of equilibria in models with limited commitment and (possible) default at equilibrium? As a first step to provide an answer to this question it is natural to investigate whether debt constraints that have been proposed in models with full commitment can be compatible with equilibrium existence in models with limited commitment. The paper is an attempt to address this issue. It shows that finitely effective debt constraints, similar to those proposed by Levine and Zame (1996) in environments with full commitment, ensure equilibrium existence in the models of Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) where commitment is limited.

A direct adaptation of finitely effective debt constraints à la Levine and Zame (1996) in those environments does not help to control debt along time. The reason is that when commitment is limited, an agent can always satisfy his budget restrictions having access to financial markets for a finite number of periods. He can do this by simply defaulting on his promises. Therefore, requiring finite-time solvency à la Levine and Zame (1996) does not restrict budget sets. In particular, it does not exclude Ponzi schemes. We address this issue by modifying appropriately the definition of finitely effective debt constraints to encompass economies with limited commitment and (possible) default at equilibrium. Working in this direction, we impose debt constraints by introducing in the setting of Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of actions with *finite equivalent payoffs*.

An interesting finding is that there is a close relation between our pro-

⁶An enforcement mechanism is said effective if it entails payments besides the value of the collateral at all nodes of a subtree.

posed budget sets and the budget sets of Levine and Zame (1996) as well as the budget sets defined through collateral obligations and no additional punishments (Araujo, Páscoa and Torres-Martínez (2002) and Kubler and Schmedders (2003)). First, our proposed debt constraints provide a natural formulation of Levine and Zame (1996) solvency requirement in those models. When there is full commitment (and payments are fully enforced) our concept of plans with finitely equivalent payoffs coincides with the concept of plans with finitely effective debt introduced by Levine and Zame (1996). Second and most important, we show that the budget feasible plans in economies with a collateral structure and zero default penalties have finite equivalent payoffs and vice versa. In other words, when there are collateral requirements but no default penalties, our budget set coincides with the standard one defined in Araujo, Páscoa and Torres-Martínez (2002) and Kubler and Schmedders (2003). This equivalence is valid for any price process (i.e., not only at equilibrium but also on out of equilibrium paths) and illustrates the hidden relation between finitely effective debt constraints and collateral requirements.

Our approach to debt constraints is certainly not the only one possible. Instead of adapting the restrictions proposed by Levine and Zame (1996), one may follow another route by considering restrictions in the spirit of Magill and Quinzii (1994) or Hernández and Santos (1996). However, it is not clear whether those borrowing constraints would be innocuous in models with collateral requirements and zero default penalties as it is the case for the constraints we propose. In that respect, we believe that modifying the approach of Levine and Zame (1996) to control debt is more suitable for models with limited commitment and collateral requirements.

Our main objective is to identify borrowing constraints that are compatible with equilibrium and at the same time allow for as much risk-sharing as possible. One may also be concerned with another issue: how difficult is to implement these borrowing constraints in a context of anonymous and competitive markets?⁷ In the context of full commitment, Magill and Quinzii (1994) identify two possible interpretations of their borrowing constraints: an objective interpretation where an external agent (an agency) has the ability to restrict agents to choose plans satisfying the borrowing constraints, and a subjective one where agents restrict themselves to satisfy these constraints. In our context, it is possible to provide an interpretation that is partly objective (market based) since it requires the presence of an agency, and partly subjective (self-monitoring) since it is the agents who restrict their trading strategies. Our interpretation is as follows: at any time period each agent conceives that he may be restricted (by an agency or by “the market”) to

⁷This question has not been addressed by Levine and Zame (1996).

issue debt for a finite number of periods. The presence of such a threat makes agents adjust their trading strategies in such a way that, if they only have access to borrowing for a finite number of periods, they can modify their consumption and investment plans so that the the loss of utility, compared to what was initially planned, is minimal. It is important to observe that the agency does not need to know agents' characteristics (in particular utility functions and default penalties). The mere fact that agents believe that the agency has the legal authority to exclude them from future credit rules out Ponzi schemes and leads to an equilibrium.

The paper is structured as follows. In Section 2 we set out the model, introduce notation, assumptions and the equilibrium concept in the absence of borrowing constraints. In Section 3 we present and discuss the new debt constraints we impose on budget feasible plans. We also introduce an equilibrium concept associated with those constraints and highlight its relation with the equilibrium concepts introduced by Levine and Zame (1996) and Araujo, Páscoa and Torres-Martínez (2002). Section 4 proves the existence of what we term *equilibrium with finite equivalent payoffs* under a mild condition on default penalties.

2 The Model

The model is essentially the one developed in Araujo, Páscoa and Torres-Martínez (2002) and extended by Páscoa and Seghir (2009) to allow for the possibility of linear default penalties. It can also be seen as an infinite horizon extension of the model proposed by Dubey, Geanakoplos and Shubik (2005).

2.1 Uncertainty and time

Let $\mathcal{T} \equiv \{0, 1, \dots, t, \dots\}$ denote the set of time periods and let S be a (infinite) set of states of nature. The available information at period $t \in \mathcal{T}$ is the same for each agent and is described by a finite partition P_t of S . Information is revealed along time, i.e., the partition P_{t+1} is finer than P_t for every t . Every pair (t, σ) where σ is a set in P_t is called a node. The set of all nodes is denoted by D and is called the event tree. We assume that there is no information at $t = 0$ and we denote by $\xi_0 = (0, S)$ the initial node. If $\xi = (t, \sigma)$ belongs to the event tree, then t is denoted by $t(\xi)$. We say that $\xi' = (t', \sigma')$ is a successor of $\xi = (t, \sigma)$ if $t' \geq t$ and $\sigma' \subset \sigma$; we use the notation $\xi' \geq \xi$. We denote by ξ^+ the set of immediate successors defined by

$$\xi^+ \equiv \{\xi' \in D : t(\xi') = t(\xi) + 1\}.$$

Because P_t is finer than P_{t-1} for every $t > 0$, for a given node $\xi \neq \xi_0$, there is a unique node ξ^- in D such that ξ is an immediate successor of ξ^- . Given a period $t \in \mathcal{T}$ we let $D_t \equiv \{\xi \in D : t(\xi) = t\}$ denote the set of nodes at period t . The set of nodes up to period t is denoted $D^t \equiv \{\xi \in D : t(\xi) \leq t\}$.

2.2 Agents and commodities

There exists a finite set L of commodities available for trade at every node $\xi \in D$. We interpret $x(\xi) \in \mathbb{R}_+^L$ as a claim to consumption at node ξ . We also write $\mathbf{1}_{\{\ell\}} \in \mathbb{R}_+^L$ for the commodity bundle consisting of one unit of commodity $\ell \in L$ and nothing else. We depart from the usual intertemporal models by allowing for some commodities to be *non-perishable*, that is, we allow for storable and durable goods as well as for commodities that may serve as physical assets (i.e., Lucas trees). Transformation of commodities is represented by a family $(Y(\xi))_{\xi \in D}$ of linear functionals $Y(\xi)$ from \mathbb{R}_+^L to \mathbb{R}_+^L . The bundle $Y(\xi)z(\xi^-)$ represents what is obtained at node ξ if the bundle $z(\xi^-) \in \mathbb{R}_+^L$ is purchased at node ξ^- . We say that the commodity ℓ is perishable at node ξ^- if $Y(\xi)\mathbf{1}_{\{\ell\}}$ is the zero vector in \mathbb{R}_+^L , and non-perishable otherwise. At each node there are spot markets for trading every commodity. We let $p = (p(\xi))_{\xi \in D}$ be the spot price process where $p(\xi) = (p(\xi, \ell))_{\ell \in L} \in \mathbb{R}_+^L$ is the price vector at node ξ .

There is a finite set I of infinitely lived agents. Each agent $i \in I$ is characterized by an endowment process $\omega^i = (\omega^i(\xi))_{\xi \in D}$ where $\omega^i(\xi) = (\omega^i(\xi, \ell))_{\ell \in L}$ is a vector in \mathbb{R}_+^L representing the endowment available at node ξ . Each agent chooses a consumption process $x = (x(\xi))_{\xi \in D}$ where $x(\xi) \in \mathbb{R}_+^L$. We denote by X the set of consumption processes. The utility function $U^i : X \rightarrow [0, +\infty]$ is assumed to be additively separable, i.e.,

$$U^i(x) \equiv \sum_{\xi \in D} u^i(\xi, x(\xi))$$

where $u^i(\xi, \cdot) : \mathbb{R}_+^L \rightarrow [0, \infty)$.

2.3 Assets and collateral

There is a finite set J of short-lived real financial assets available for trade at each node. For each asset j , the bundle yielded at node ξ is denoted by $A(\xi, j) \in \mathbb{R}_+^L$. We let $q = (q(\xi))_{\xi \in D}$ be the asset price process where $q(\xi) = (q(\xi, j))_{j \in J} \in \mathbb{R}_+^J$ represents the asset price vector at node ξ . We denote by $\theta^i(\xi) \in \mathbb{R}_+^J$ the vector of purchases and by $\varphi^i(\xi) \in \mathbb{R}_+^J$ the vector of short-sales at each node ξ .

Following the seminal contribution of Geanakoplos (1997) and Geanakoplos and Zame (2002) for finite horizon models, and Araujo, Páscoa and Torres-Martínez (2002) together with Páscoa and Seghir (2009) for infinite horizon models, assets are collateralized in the sense that for every unit of asset j sold at a node ξ , agents should buy a collateral bundle $C(\xi, j) \in \mathbb{R}_+^L$ that protects lenders in case of default. We assume that payments can be enforced only through the seizure of the collateral. At a node ξ , agent i should deliver the promise $V(p, \xi)\varphi^i(\xi^-)$ where

$$V(p, \xi) = (V(p, \xi, j))_{j \in J} \quad \text{and} \quad V(p, \xi, j) \equiv p(\xi)A(\xi, j).$$

However, agent i may decide to default and choose a delivery $d^i(\xi, j)$ in units of account. Since the collateral can be seized, this delivery must satisfy

$$d^i(\xi, j) \geq D(p, \xi, j)\varphi^i(\xi^-, j)$$

where

$$D(p, \xi, j) \equiv \min\{p(\xi)A(\xi, j), p(\xi)Y(\xi)C(\xi^-, j)\}.$$

Remark 2.1. Kubler and Schmedders (2003) propose a model where the collateral requirements are imposed in terms of physical assets. We show that their model is a particular case of the model proposed by Araujo, Páscoa and Torres-Martínez (2002). In that respect whenever we are referring to the model proposed by Araujo, Páscoa and Torres-Martínez (2002) we are also referring to the one proposed by Kubler and Schmedders (2003).

If there is a specific commodity $g \in L$ satisfying the following properties, then this commodity can be interpreted as a physical asset or a Lucas tree.

(i) At initial node ξ_0 , each agent i has an initial endowment $\omega^i(\xi_0, g) \geq 0$ of commodity g which represents his share of the tree. At subsequent nodes $\xi > \xi_0$, agent i has no initial endowment in commodity g .

(ii) One unit of commodity g purchased at node ξ delivers at node $\mu \in \xi^+$ the bundle

$$y(\mu) \equiv Y(\mu)\mathbf{1}_{\{g\}} \in \mathbb{R}_+^L.$$

The g -th coordinate $y(\mu, g)$ is equal to 1, i.e., the physical asset is long lived.

(iii) Each agent i is indifferent with respect to commodity g , i.e., for each agent $i \in I$, for each node $\xi \in D$, for each consumption bundle $c \in \mathbb{R}_+^L$, we have

$$u^i(\xi, c + \mathbf{1}_{\{g\}}) = u^i(\xi, c).$$

(iv) In every successor node $\mu \in \xi^+$, the transformed bundle of one unit of commodity g purchased at any node ξ , is a desirable bundle, i.e., $y(\mu)$ is a bundle in \mathbb{R}_+^L such that for each consumption bundle $c \in \mathbb{R}_+^L$, we have⁸

$$u^i(\mu, c + y(\mu)) > u^i(\mu, c).$$

If at every node $\xi \in D$, the collateral bundle $C(\xi, j)$ is only in terms of commodity g , then the collateral structure of our model (and the one in Araujo, Páscoa and Torres-Martínez (2002) and Páscoa and Seghir (2009)) reduces to the one considered by Kubler and Schmedders (2003).

Following Dubey, Geanakoplos and Shubik (1990) (and Dubey, Geanakoplos and Shubik (2005)), we assume that agent i feels a disutility $\lambda^i(\xi, j) \in [0, +\infty]$ from defaulting.⁹ More precisely, if an agent defaults at node ξ , then he suffers at $t = 0$, the disutility

$$\sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)]^+}{p(\xi)v(\xi)}$$

where $(v(\xi))_{\xi \in D}$ is an exogenously specified process in \mathbb{R}_{++}^L that is uniformly bounded away from 0.¹⁰ In that case, agent i may have an incentive to deliver more than the minimum between his debt and the depreciated value of his collateral, i.e., we may have $d^i(\xi, j) > D(p, \xi, j)\varphi^i(\xi^-, j)$.

As in Dubey, Geanakoplos and Shubik (2005) assets are thought as pools. At each node ξ the sales $\varphi^i(\xi, j)$ are pooled at the market for asset j . The deliveries $d^i(\xi, j)$ on asset j are also pooled and the buyers of pool j receive a pro rata share of all its different sellers' deliveries. We assume that lenders rationally anticipate that every borrower delivers at least $D(p, \xi, j)$ on each unit asset j sold at node ξ^- . Therefore, agents anticipate that each share of pool j delivers a fraction $V(\kappa, p, \xi, j)$ of its promise $V(p, \xi, j)$ defined by

$$V(\kappa, p, \xi, j) = \kappa(\xi, j)V(p, \xi, j) + (1 - \kappa(\xi, j))D(p, \xi, j)$$

where $\kappa(\xi, j) \in [0, 1]$ will be determined at equilibrium such that deliveries match payments.¹¹ The buyer of asset j does not need to know the identities of

⁸Since each agent i is indifferent with respect to commodity g , the bundle delivered by the tree must satisfy $y(\mu, \ell) > 0$ for at least one commodity $\ell \neq g$.

⁹Models with non-pecuniary penalties for default also include Diamond (1984), Rea (1984), who considers contracts involving "arm-breaking", Zame (1993), Araujo, Monteiro and Páscoa (1998), Bisin and Gottardi (1999), Santos and Scheinkman (2001), Lacker (2001) and Páscoa and Seghir (2009).

¹⁰More precisely, we assume that there exists $\underline{v} > 0$ such that for every node $\xi \in D$ and every commodity $\ell \in L$, we have $v(\xi, \ell) \geq \underline{v}$.

¹¹If all the sellers of asset j at node ξ^- fully deliver on their promises at the successor node ξ then $\kappa(\xi, j) = 1$, while if all sellers fully default on their promises then $\kappa(\xi, j) = 0$.

the sellers or the quantities of their sales. All that matters to him is the price $q(\xi, j)$ of one unit of asset and the anticipated delivery rates $(\kappa(\mu, j))_{\mu \in \xi^+}$.

2.4 Budget set without debt constraints

We let A be the space of adapted processes $a = (a(\xi))_{\xi \in D}$ with¹²

$$a(\xi) = (x(\xi), \theta(\xi), \varphi(\xi), d(\xi)) \in \mathbb{R}_+^L \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^J.$$

Given a process (p, q, κ) of commodity prices, asset prices and delivery rates, agent i 's choice $a^i = (x^i, \theta^i, \varphi^i, d^i) \in A$ must satisfy, in each decision node $\xi \in D$, the following constraints:

(a) solvency constraint:

$$\begin{aligned} p(\xi)x^i(\xi) + \sum_{j \in J} d^i(\xi, j) + q(\xi)\theta^i(\xi) \\ \leq p(\xi)[\omega^i(\xi) + Y(\xi)x^i(\xi^-)] + V(\kappa, p, \xi)\theta^i(\xi^-) + q(\xi)\varphi^i(\xi); \end{aligned} \quad (2.1)$$

(b) collateral requirement:

$$C(\xi)\varphi^i(\xi) \leq x^i(\xi); \quad (2.2)$$

(c) minimum delivery:

$$\forall j \in J, \quad D(p, \xi, j)\varphi^i(\xi^-, j) \leq d^i(\xi, j). \quad (2.3)$$

The set of plans $a = (x, \theta, \varphi, d) \in A$ satisfying constraints (2.1), (2.2) and (2.3) is called the (unconstrained) budget set and is denoted by $B^i(p, q, \kappa)$.

2.5 The payoff function

Consider that agent i has chosen the plan $a = (x, \theta, \varphi, d)$ under a process of prices and delivery rates $\pi = (p, q, \kappa)$. He enjoys the utility

$$U^i(x) = \sum_{\xi \in D} u^i(\xi, x(\xi)) \in [0, \infty]$$

but he suffers the disutility $W^i(p, a) \in [0, \infty]$ defined by

$$W^i(p, a) \equiv \sum_{\xi > \xi_0} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

¹²By convention we pose $a(\xi_0^-) = (x(\xi_0^-), \theta(\xi_0^-), \varphi(\xi_0^-), d(\xi_0^-)) = (0, 0, 0, 0)$.

We would like to define the payoff $\Pi^i(p, a)$ of the plan a as the following difference

$$\Pi^i(p, a) = U^i(x) - W^i(p, a).$$

Unfortunately, $\Pi^i(p, a)$ may not be well defined if both $U^i(x)$ and $W^i(p, a)$ are infinite. We propose to consider the binary relation $\succ_{i,p}$ defined on A by

$$\tilde{a} \succ_{i,p} a \iff \exists \varepsilon > 0, \quad \exists T \in \mathbb{N}, \quad \forall t \geq T, \quad \Pi^{i,t}(p, \tilde{a}) \geq \Pi^{i,t}(p, a) + \varepsilon$$

where

$$\Pi^{i,t}(p, a) \equiv U^{i,t}(x) - W^{i,t}(p, a), \quad U^{i,t}(x) \equiv \sum_{\xi \in D^t} u^i(\xi, x(\xi))$$

and

$$W^{i,t}(p, a) \equiv \sum_{\xi \in D^t \setminus \{\xi_0\}} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

According to this definition, a plan \tilde{a} is strictly preferred to a if the difference of payoffs $\Pi^{i,t}(p, \tilde{a}) - \Pi^{i,t}(p, a)$ between the two plans is uniformly strictly positive for every period t large enough.¹³

Observe that if $\Pi^i(p, \tilde{a})$ and $\Pi^i(p, a)$ are finite then $\tilde{a} \succ_{i,p} a$ if and only if $\Pi^i(p, \tilde{a}) > \Pi^i(p, a)$. We denote by $\text{Pref}^i(p, a)$ the set of plans $\tilde{a} \succ_{i,p} a$ strictly preferred to plan a by agent i .

2.6 Assumptions

For each agent i , we denote by Ω^i the process of accumulated endowments, defined recursively by $\Omega^i(\xi) = Y(\xi)\Omega^i(\xi^-) + \omega^i(\xi)$ where $\Omega^i(\xi_0) = \omega^i(\xi_0)$. The process $\sum_{i \in I} \Omega^i$ of accumulated aggregate endowments is denoted by Ω . The following assumptions on the characteristics of the economy are standard in the literature of infinite horizon models with collateral requirements.

Assumption 2.1 (Agents). For every agent i ,

(H.1) the process of accumulated endowments is strictly positive and uniformly bounded from above, i.e.,

$$\exists \bar{\Omega}^i \in \mathbb{R}_{++}^L, \quad \forall \xi \in D, \quad \Omega^i(\xi) \in \mathbb{R}_{++}^L \quad \text{and} \quad \Omega^i(\xi) \leq \bar{\Omega}^i;$$

¹³The sequence of differences $(\Pi^{i,t}(p, \tilde{a}) - \Pi^{i,t}(p, a))_{t \geq 1}$ need not be converging.

(H.2) for every node ξ , the utility function $u^i(\xi, \cdot)$ is concave, continuous and strictly increasing,¹⁴ with $u^i(\xi, 0) = 0$;

(H.3) the infinite sum $U^i(\Omega)$ is finite.

Assumption 2.2 (Financial assets). For every asset j and node ξ , the collateral $C(\xi, j)$ is not zero.

It should be clear that these assumptions always hold throughout the paper.

2.7 Equilibrium without debt constraints

We denote by Ξ the set of prices and delivery rates (p, q, κ) normalized as follows¹⁵

$$\forall \xi \in D, \quad (p(\xi), q(\xi)) \in \Delta(L \times J), \quad p(\xi) \in \mathbb{R}_{++} \quad \text{and} \quad \kappa(\xi) \in [0, 1]^J. \quad (2.4)$$

Given a process (p, q, κ) of commodity prices, asset prices and delivery rates, we denote by $d^i(p, q, \kappa)$ the demand set defined by

$$d^i(p, q, \kappa) \equiv \{a \in B^i(p, q, \kappa) : \text{Pref}^i(p, a) \cap B^i(p, q, \kappa) = \emptyset\}.$$

Definition 2.1. A *competitive equilibrium* for the economy \mathcal{E} is a family of prices and delivery rates $(p, q, \kappa) \in \Xi$ and an allocation $\mathbf{a} = (a^i)_{i \in I}$ with $a^i \in A$ such that

(a) for every agent i , the plan a^i is optimal, i.e.,

$$a^i \in d^i(p, q, \kappa);$$

(b) commodity markets clear at every node, i.e.,

$$\sum_{i \in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \quad (2.5)$$

and for all $\xi \neq \xi_0$,

$$\sum_{i \in I} x^i(\xi) = \sum_{i \in I} [\omega^i(\xi) + Y(\xi)x^i(\xi^-)]; \quad (2.6)$$

¹⁴Assuming that the function $u^i(\xi, \cdot)$ is strictly increasing is not compatible with the interpretation of a commodity as a Lucas tree. This assumption was made only for expositional purposes and can be weakened as follows: for every ξ the function $u^i(\xi, \cdot)$ is non-decreasing and there exists a commodity ℓ that is strictly desirable in the sense that for every pair x, y in \mathbb{R}_+^L , we have $u^i(\xi, x + y) > u^i(\xi, x)$ provided that $y(\ell) > 0$.

¹⁵The pair $(p(\xi), q(\xi))$ belongs to the simplex $\Delta(L \times J)$ if $p(\xi) \in \mathbb{R}_+^L$, $q(\xi) \in \mathbb{R}_+^J$ and $\sum_{\ell \in L} p(\xi, \ell) + \sum_{j \in J} q(\xi, j) = 1$.

(c) asset markets clear at every node, i.e., for all $\xi \in D$,

$$\sum_{i \in I} \theta^i(\xi) = \sum_{i \in I} \varphi^i(\xi); \quad (2.7)$$

(d) deliveries match at every node, i.e., for all $\xi \neq \xi_0$ and all $j \in J$,

$$\sum_{i \in I} V(\kappa, p, \xi, j) \theta^i(\xi^-, j) = \sum_{i \in I} d^i(\xi, j). \quad (2.8)$$

The set of allocations $\mathbf{a} = (a^i)_{i \in I}$ in A satisfying the market clearing conditions (2.5), (2.6) and (2.7) is denoted by F . Each allocation in F is called physically feasible. A plan $a^i \in A$ is called physically feasible if there exists a physically feasible allocation \mathbf{b} such that $a^i = b^i$. The set of physically feasible plans is denoted by F^i . We denote by $\text{Eq}(\mathcal{E})$ the set of competitive equilibria for the economy \mathcal{E} .

3 Debt constraints

In this section, we show how to adapt the *finitely effective* debt constraints proposed by Levine and Zame (1996) to infinite horizon models with limited commitment and default penalties. While keeping the minimal ability to borrow and lend that we expect in our model, we prove that the proposed constraints are compatible with equilibrium (precluding agents to run Ponzi schemes). Moreover, our constraints appear to have an additional appealing feature: we show that the budget sets associated with those constraints coincide with the standard budget sets of economies having a collateral structure but no penalties (as defined in Araujo, Páscoa and Torres-Martínez (2002) and Kubler and Schmedders (2003)).

3.1 Infinite default penalties

When default penalties are infinite and the collateral requirements are zero, our model reduces to the one studied by Magill and Quinzii (1994) and Levine and Zame (1996). In the absence of debt constraints, an equilibrium may not exist: all traders would attempt to finance unbounded levels of consumption by unbounded levels of borrowing. To rule out Ponzi schemes, Levine and Zame (1996) (see also Levine and Zame (2002)) formalize the concept of plans with *finitely effective* debt by requiring agents' actions to be budget compatible with the threat that, at any period, agents may be restricted to have access to borrowing for only a finite number of periods. In other words, an agent's

debt is finitely effective if at any period, the debt is repayable within a finite horizon. More formally, we consider the following definition due to Levine and Zame (1996).

Definition 3.1. A plan $a \in B^i(p, q, \kappa)$ is said to have *finitely effective debt*, if for each period $t \geq 0$, there exists a period $T > t$ and a plan \hat{a} also in the budget set $B^i(p, q, \kappa)$ such that

(i) up to period t both plans coincide, i.e.,

$$\forall \xi \in D^t, \quad \hat{a}(\xi) = a(\xi);$$

(ii) at every node after period T , there is solvency without borrowing, i.e.,

$$\forall \xi \in D, \quad t(\xi) \geq T \implies \hat{\varphi}(\xi) = 0.$$

The intuition behind Definition 3.1 can be better understood if we think about the role of those restrictions in the finite horizon framework. No short selling at the terminal date implicitly imposes a solvency requirement at earlier dates. That is, at any node agents should hold an amount of debt that they will be able to repay by the end of the terminal date. In the absence of a terminal date, it is necessary to impose explicitly or implicitly that solvency requirement.

Remark 3.1. Consider the following notation. For each period t , we denote by A^t the set of plans $a \in A$ where $a(\xi) = (0, 0, 0, 0)$ for each ξ such that $t(\xi) > t$. If a is a plan in A and t is a period, we denote by $a\mathbf{1}_{[0,t]}$ the plan in A^t which coincides with a for every node $\xi \in D^t$.¹⁶ Following this notation, a plan a has a finitely effective debt if for each period $t \geq 0$, there exists a subsequent period $T > t$ and a plan \hat{a} such that

$$\hat{a} \in B^i(p, q, \kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \hat{a}\mathbf{1}_{[0,t]}$$

where C^T is the set of plans a in A without borrowing after period T in the sense that

$$\forall \xi \in D, \quad t(\xi) \geq T \implies \hat{\varphi}(\xi) = 0.$$

Instead of restricting plans to be finitely effective, one may consider the following alternative restriction.

Definition 3.2. A budget feasible plan $a \in B^i(p, q, \kappa)$ is said to have *finite equivalent utility* when for every period $t \geq 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan \hat{a} such that

¹⁶The plan $a\mathbf{1}_{[0,t]}$ can be interpreted as a “truncation” of a up to period t .

- (i) the plans a and \hat{a} coincide up to period t , i.e., $a\mathbf{1}_{[0,t]} = \hat{a}\mathbf{1}_{[0,t]}$;
- (ii) the plan \hat{a} is budget feasible and there is no borrowing after period T , i.e., $\hat{a} \in B^i(p, q, \kappa) \cap C^T$;
- (iii) the utility of the plan \hat{a} may be lower than the payoff of a but not more than ε , i.e.,

$$\inf_{\tau \geq T} [U^{i,\tau}(p, \hat{a}) - U^{i,\tau}(p, a)] \geq -\varepsilon.$$

In other words, a budget feasible plan a has finite equivalent utility if in case where at some period t the agent is restricted to have access to borrowing for finitely many periods, then he can find an alternative plan \hat{a} doing the job, i.e., satisfying (i) and (ii); but at the same time the utility loss can be made as small as desired.

The following proposition provides an equivalence between plans with finitely effective debt and plans having finite equivalent utility. This alternative characterization will be proven particularly useful in the process of modifying finitely effective constraints to encompass models with limited commitment.

Proposition 3.1. Assume that the default penalty is infinite and consider a budget feasible plan $a \in B^i(p, q, \kappa)$ with a finite utility $U^i(x) < \infty$. The plan a has a finitely effective debt, if and only if, it has finite equivalent utility.

Proof of Proposition 3.1. Let $a \in B^i(p, q, \kappa)$ be a budget feasible plan with a finite utility $U^i(x) < \infty$. It is obvious that if a has finite equivalent utility, then it has a finitely effective debt. The converse deserves more attention. Assume that the plan a has a finitely effective debt. Fix a period t and $\varepsilon > 0$. If we apply the definition to the period t , we get the existence of a period $T > t$ and a plan \hat{a} such that

$$\hat{a} \in B^i(p, q, \kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \hat{a}\mathbf{1}_{[0,t]}.$$

Unfortunately, we do not know if $U^{i,T}(\hat{a}) \geq U^{i,T}(x) - \varepsilon$. However, we know that the utility $U^i(x)$ is finite. Therefore, there exists $t' > t$ such that

$$\sum_{s > t'} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \leq \varepsilon. \tag{3.1}$$

Now, applying the definition of *finitely effective debt* for the period t' , there exists a period $T > t'$ and a plan \hat{a} such that

$$\hat{a} \in B^i(p, q, \kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t']} = \hat{a}\mathbf{1}_{[0,t']}.$$

Now fix $\tau \geq T$. Since $T > t'$, we have

$$U^{i,\tau}(\hat{x}) \geq U^{i,T}(\hat{x}) \geq U^{i,t'}(\hat{x}) = U^{i,t'}(x) \geq U^{i,\tau}(x) - \sum_{t' < s \leq \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)).$$

It follows from (3.1) that $U^{i,\tau}(\hat{x}) \geq U^{i,\tau}(x) - \varepsilon$. \square

3.2 Finite default penalties

The concept of finitely effective debt constraints makes perfect sense in models with full enforcement and perfect commitment (i.e., no default). However, with limited commitment, imposing finitely effective debt constraints does not help to control debt along time. We provide an explanation below. Let $a = (x, \theta, \varphi, d)$ be a plan in $B^i(p, q, \kappa)$ and t be any period. Consider the plan \hat{a} defined by

$$\forall \xi \in D, \quad \hat{a}(\xi) = \begin{cases} a(\xi) & \text{if } t(\xi) \leq t \\ (\omega^i(\xi), 0, 0, D(p, \xi)\varphi(\xi^-)) & \text{if } t(\xi) = t + 1 \\ (\omega^i(\xi), 0, 0, 0) & \text{if } t(\xi) > t + 1. \end{cases}$$

This plan belongs to the set $B^i(p, q, \kappa) \cap C^{t+1}$ and coincides with a on every node up to period t . That is, under limited commitment, any plan $a \in B^i(p, q, \kappa)$ has finitely effective debt according to Definition 3.1. Agents can always default up to the minimum value between their debt and the depreciated value of their collateral. Therefore, there is no hope to bound debt along time.

We introduce hereafter an endogenous restriction on trades that allows to encompass models with limited commitment and finite default penalties. The point of our departure is Proposition 3.1 where it is shown that, when default penalties are infinite, restricting plans to have finitely effective debt is equivalent to restricting plans to have finite equivalent utility. This equivalence breaks down in the presence of finite default penalties. In this case, we proceed by replacing ‘‘utility’’ by ‘‘payoff’’ and we introduce the concept of plans with *finite equivalent payoffs*. We claim that requiring plans to have finite equivalent payoffs provides an appropriate adaptation of finitely effective debt constraints to models with limited commitment and finite default penalties. The formal definition is as follows.

Definition 3.3. A plan a in the budget set $B^i(p, q, \kappa)$ has *finite equivalent payoffs* if for every period $t \geq 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan \hat{a} such that

- (i) the plans a and \hat{a} coincide up to period t , i.e., $a\mathbf{1}_{[0,t]} = \hat{a}\mathbf{1}_{[0,t]}$;
- (ii) the plan \hat{a} is budget feasible and there is no borrowing after period T , i.e. $\hat{a} \in B^i(p, q, \kappa) \cap C^T$;
- (iii') the payoff of the plan \hat{a} may be lower than the payoff of the initial plan a but not more than ε , i.e.,

$$\inf_{\tau \geq T} [\Pi^{i,\tau}(p, \hat{a}) - \Pi^{i,\tau}(p, a)] \geq -\varepsilon.$$

The interpretation of a plan with finite equivalent payoff is similar to the one of a plan of finite equivalent utility. The only difference is that we replace “utility” by “payoff”. This is very intuitive since agents may suffer a loss in utility when defaulting.

3.3 Implementation issues

The introduction of debt constraints raises issues related to the implementation of those constraints in decentralized anonymous markets. We provide hereafter an interpretation of our proposed debt constraints that is partly objective (market based) since it requires the presence of an agency, and partly subjective (self-monitoring) since it is the agents who restrict their trading strategies.

When making a plan a^i , we assume that agent i conceives that, at any period t , there is a possibility that an agency will not allow him to have access to borrowing forever.¹⁷ We also assume that agents have no model in mind in order to compute the “true” probability of the agency’s intervention. They simply believe that it may happen at any time period, contingent to any history of shocks on the primitives (the tree D). The agent may believe that he will be able to negotiate with the agency on the number of periods he will still continue having access to borrowing.

Assume for the moment that agents are not restricted when they choose their portfolio plans. When making those plans, agents should modify the initial description of uncertainty provided by the tree D by adding states of nature representing the agency’s intervention. If agents are assumed to have a strong ambiguity aversion (recall that they have no prior about the realization of those additional states) it is reasonable to assume that they will use a maxmin criterium (as in Gilboa and Schmeidler (1989)) to evaluate those

¹⁷We do not explicitly model the reasons why an agency would decide at some point in time to restrict the trading strategies of a specific agent. For instance, one may assume that an exogenous shock may affect the agency’s behavior.

plans. In this case, making plans contingent to the additional states of nature (representing the agency's decision to intervene) when there are no constraints on financial trades and using a maxmin criterium to rank them is equivalent to making plans that are only contingent to shocks on primitives (the tree D) and are restricted to have finitely equivalent payoffs. This is because, in the bad situation where the agency restricts the access to credit markets in some period t , agent i can deviate from the initially chosen plan a^i by following another budget feasible plan \hat{a}^i , different from a^i only after period t and consistent with the restriction that borrowing occurs only for a finite number of periods after t , whose payoff is as close as desired to the payoff agent i would have enjoyed with the initial plan a^i .

One could alternatively argue for a pure market based interpretation where the central authority enforces the borrowing limits at any node. We think that our interpretation is more appealing from an economic point of view. This is because it only requires the central authority to have the legal power to exclude an agent from borrowing, while a pure market based interpretation requires a central authority that knows the agents' characteristics.¹⁸

3.4 Equilibrium with finite equivalent payoffs

We denote by $B_\star^i(p, q, \kappa)$ the set of all plans in $B^i(p, q, \kappa)$ having finite equivalent payoffs.

Definition 3.4. A *competitive equilibrium with finite equivalent payoffs* for the economy \mathcal{E} is a family of prices and delivery rates $(p, q, \kappa) \in \Xi$ together with an allocation $\mathbf{a} = (a^i)_{i \in I}$ with $a^i \in A$ such that the conditions of market clearing (b), (c) and (d) in Definition 2.1 are satisfied and the unconstrained optimality condition (a) is replaced by

(a') for every agent i , the plan a^i has finite equivalent payoffs and is optimal among all budget feasible plans with finite equivalent payoffs, i.e.,

$$a^i \in d_\star^i(p, q, \kappa) \equiv \{a \in B_\star^i(p, q, \kappa) : \text{Pref}^i(p, a) \cap B_\star^i(p, q, \kappa) = \emptyset\}.$$

We denote by $\text{Eq}_\star(\mathcal{E})$ the set of competitive equilibria with finite equivalent payoffs for the economy \mathcal{E} . We will prove in Section 4.2 that set $\text{Eq}_\star(\mathcal{E})$ is

¹⁸An objective or subjective interpretation of participation constraints à la Kehoe and Levine (1993) (see also Zhang (1997), Alvarez and Jermann (2000) and Kehoe and Levine (2001)) requires the agency or the agents to know the other agents' characteristics. For instance, under self-monitoring, lenders will only provide credit to the extent that they are able to calculate the borrowers' expected discounted lifetime utility from participating in the asset markets and their corresponding utility in autarky.

non-empty under a mild condition on default penalties. Before addressing the existence issue, we explore hereafter the relation between the equilibrium concept that we have just introduced with the one found in Araujo, Páscoa and Torres-Martínez (2002).

3.5 No default penalty

We consider the case where collateral repossession is the only enforcement mechanism and that default penalties are equal to zero as in Araujo, Páscoa and Torres-Martínez (2002) and Kubler and Schmedders (2003). One may expect $B_\star^i(p, q, \kappa)$ to be a strict subset of $B^i(p, q, \kappa)$. However, as the following proposition shows, the two sets coincide. In fact, in the model proposed by Araujo, Páscoa and Torres-Martínez (2002), any budget feasible allocation with a finite utility has finite equivalent payoffs. This is a consequence of the absence of default penalties or explicit economic punishments.

Proposition 3.2. Assume that there is no default penalty and let $a = (x, \theta, \varphi, d)$ be a plan in the budget set $B^i(p, q, \kappa)$. If $U^i(x)$ is finite then a has finite equivalent payoffs, i.e., a belongs to $B_\star^i(p, q, \kappa)$.

Proof of Proposition 3.2. Fix an agent i and consider a budget feasible plan $a \in B^i(p, q, \kappa)$ with a finite utility. Fix a period $t \geq 1$ and $\varepsilon > 0$. Since $U^i(x)$ is finite, there exists $T \geq t + 1$ such that

$$\sum_{\tau \geq T} \sum_{\xi \in D_\tau} u^i(\xi, x(\xi)) \leq \varepsilon.$$

Consider now the plan \hat{a} defined by

$$\hat{a}(\xi) = \begin{cases} a(\xi) & \text{if } t(\xi) < T \\ (\omega^i(\xi), 0, 0, \hat{d}(\xi)) & \text{if } t(\xi) = T \\ (\omega^i(\xi), 0, 0, 0) & \text{if } t(\xi) > T \end{cases}$$

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \hat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Observe that the plan \hat{a} is budget feasible, belongs to C^T and satisfies

$$\hat{a}\mathbf{1}_{[0, T-1]} = a\mathbf{1}_{[0, T-1]}.$$

Fix $\tau \geq T$. Since $T - 1 \geq t$, in order to prove that the plan a has finite equivalent payoffs, we need to compare $U^{i,\tau}(\hat{x})$ and $U^{i,\tau}(x)$. Observe that

$$\begin{aligned}
U^{i,\tau}(\hat{x}) &= U^{i,T-1}(x) + \sum_{T \leq s \leq \tau} \sum_{\xi \in D_s} u^i(\xi, \omega^i(\xi)) \\
&\geq U^{i,T-1}(x) \\
&\geq U^{i,\tau}(x) - \sum_{T \leq s \leq \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \\
&\geq U^{i,\tau}(x) - \varepsilon.
\end{aligned}$$

We have thus proved that the plan a has finite equivalent payoffs. \square

A direct implication of the last proposition is that, when there is no loss of utility in case of default, the sets $\text{Eq}(\mathcal{E})$ and $\text{Eq}_*(\mathcal{E})$ coincide. This observation allows us to obtain the existence result of Araujo, Páscoa and Torres-Martínez (2002) as a direct corollary of our equilibrium existence result (see Section 4).

Proposition 3.3. If there is no default penalty then (π, \mathbf{a}) is a competitive equilibrium, if and only if, it is a competitive equilibrium with finite equivalent payoffs, i.e., the sets $\text{Eq}(\mathcal{E})$ and $\text{Eq}_*(\mathcal{E})$ coincide.

Proof of Proposition 3.3. Let $(\pi, \mathbf{a}) \in \text{Eq}(\mathcal{E})$ be a competitive equilibrium. Fix an agent $i \in I$. In order to prove that a^i belongs to the demand $d_*^i(\pi)$, it is sufficient to prove that a^i has finite equivalent payoffs. Since \mathbf{a} is feasible we have $x^i(\xi) \leq \Omega(\xi)$. From (H.3), we get that $U^i(x^i)$ is finite. The desired result follows from Proposition 3.2.

Now let $(\pi, \mathbf{a}) \in \text{Eq}_*(\mathcal{E})$ be a competitive equilibrium with finite equivalent payoffs. We only have to prove that a^i belongs to $d^i(\pi)$ for each agent i . Fix an agent i and assume by contradiction that there exists a plan a in $B^i(\pi)$ such that $U^i(x) > U^i(x^i)$. If $U^i(x)$ is finite then, applying Proposition 3.2, we get that $a \in B_*^i(\pi)$: contradiction. Therefore, we must have $U^i(x) = \infty$, implying that there exists $T \geq 1$ such that

$$U^{i,T}(x) > U^i(x^i).$$

Consider the plan \hat{a} defined by

$$\hat{a}(\xi) = \begin{cases} a(\xi) & \text{if } t(\xi) \leq T \\ (\omega^i(\xi), 0, 0, \hat{d}(\xi)) & \text{if } t(\xi) = T + 1 \\ (\omega^i(\xi), 0, 0, 0) & \text{if } t(\xi) > T + 1 \end{cases}$$

where

$$\forall \xi \in D_{T+1}, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Since the plan \widehat{a} is budget feasible and belongs to C^{T+1} , it has finite equivalent payoffs and belongs to $B_\star^i(p, q, \kappa)$. Moreover we have

$$U^i(\widehat{x}) = U^{i,T}(x) + \sum_{\xi \in D \setminus D^T} u^i(\xi, \omega^i(\xi)) > U^i(x^i).$$

This contradicts the optimality of x^i in $B_\star^i(p, q, \kappa)$. \square

4 Precluding Ponzi schemes

Levine and Zame (1996) proved that finitely effective debt constraints are compatible with equilibrium when the default penalty is infinite and no collateral is required. We argued in the previous section that a reasonable adaptation of those endogenous borrowing constraints to models with limited commitment is to restrict plans to have finite equivalent payoffs. We formally defined the concept of equilibrium with finite equivalent payoffs and we have shown its relation with respect to the equilibrium concepts found in the papers of Araujo, Páscoa and Torres-Martínez (2002) and Kubler and Schmedders (2003). In this section, we are concerned with the issue of existence of such equilibria. We show that if agents are myopic with respect to default penalties, restricting actions to have finite equivalent payoffs allows to rule out Ponzi schemes and guarantees the existence of an equilibrium. Myopia in our setting refers to the time preference of default: the disutility of defaulting today is greater than the disutility of defaulting in the distant future and vanishes in the long run. In other words, myopia implies a reasonable restriction on the asymptotic behavior of default penalties. We exhibit below a large class of “standard” economies for which agents are myopic with respect to default penalties.

4.1 Myopia with respect to default penalties

Before introducing the formal definition of myopic agents with respect to default penalties, we need to introduce some notations. For each asset j and node ξ , we denote by $M(\xi, j)$ the real number

$$\min_{\ell \in L} \frac{\Omega(\xi, \ell)}{C(\xi, j, \ell)}$$

which corresponds to the maximum amount of short-sales in asset j at node ξ that is consistent with the equilibrium condition of market clearing. Observe

that under Assumption 2.2, we have $M(\xi, j) < \infty$. Finally, for every node $\xi \neq \xi_0$ we let¹⁹

$$H(\xi, j) = M(\xi^-, j) \sup_{p \in \Delta(L)} \frac{[pA(\xi, j) - pY(\xi)C(\xi^-, j)]^+}{pv(\xi)}.$$

The quantity $H(\xi, j)$ is the maximum amount in real terms that an agent may default on asset j if his plan is physically feasible. Indeed, it is straightforward to verify that if $a = (x, \theta, \varphi, d)$ in A is a physically feasible plan and (p, q, κ) in Π is a process of prices and delivery rates, then for each node ξ and each asset j , we have

$$\varphi(\xi, j) \leq M(\xi, j) \quad \text{and} \quad [V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+ \leq H(\xi, j).$$

Definition 4.1. Agent i is said to be *myopic with respect to default penalties* if the disutility suffered at the initial period from defaulting in the long run is negligible, i.e.,

$$\liminf_{T \rightarrow \infty} \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) H(\xi, j) = 0.$$

Assuming that agents are myopic with respect to default penalties is a very mild assumption since it is automatically satisfied for every standard economy as defined below (see e.g. Araujo and Sandroni (1999)).

Definition 4.2. The economy \mathcal{E} is said *standard* if Assumptions 2.1 and 2.2 are satisfied and if for each agent i there exist

- (S.1) a discount factor $\beta_i \in (0, 1)$;
- (S.2) a sequence $(P_t^i)_{t \geq 1}$ of beliefs about nodes at period t represented by a probability $P_t^i \in \text{Prob}(D_t)$;
- (S.3) a Bernoulli function $v^i : D \times \mathbb{R}_+^L \rightarrow [0, \infty)$ where $v^i(\xi, \cdot)$ is the cardinal felicity function at node ξ ;
- (S.4) a current default penalty $\mu^i(\xi, j) \in (0, \infty)$ for each node $\xi > \xi_0$;

such that for each node $\xi \in D$,

$$u^i(\xi, \cdot) = [\beta_i]^{t(\xi)} P_{t(\xi)}^i(\xi) v^i(\xi, \cdot)$$

for each $j \in J$,

$$\lambda^i(\xi, j) = [\beta_i]^{t(\xi)} P_{t(\xi)}^i(\xi) \mu^i(\xi, j)$$

¹⁹The set $\Delta(L)$ is the simplex in \mathbb{R}_+^L , i.e., $\Delta(L) = \{p \in \mathbb{R}_+^L : \sum_{\ell \in L} p(\ell) = 1\}$.

and the processes $(A(\xi, j))_{\xi > \xi_0}$, $(\mu^i(\xi, j))_{\xi > \xi_0}$ and $(G(\xi, j))_{\xi \in D}$ are uniformly bounded from above, where

$$G(\xi, j) = \frac{1}{\max_{\ell \in L} C(\xi, j, \ell)}.$$

Remark 4.1. In a standard economy, one may have that current default penalties are time and state independent, i.e., $\mu^i(\xi, j) = \bar{\mu}(j)$. In that case, assuming that agents are myopic with respect to default penalties does not impose any restriction on $\bar{\mu}(j)$: it can be as large as desired.

When agents are myopic with respect to default penalties, any budget and physically feasible plan $a \in B^i(p, q, \kappa) \cap F^i$ has actually finite equivalent payoffs. This result will turn out to be crucial in the process of proving the existence of an equilibrium with finite equivalent payoffs.

Proposition 4.1. If agent i is myopic with respect to default penalties, then every budget and physically feasible plan has finite equivalent payoffs. In other words, we have

$$B^i(p, q, \kappa) \cap F^i \subset B^i_*(p, q, \kappa).$$

Proof of Proposition 4.1. Fix an agent i and consider a plan a that is budget and physically feasible, i.e., $a \in B^i(p, q, \kappa) \cap F^i$. Fix a period $t \geq 1$ and $\varepsilon > 0$. Since the allocation a is physically feasible, we have $x(\xi) \leq \Omega(\xi)$, implying that

$$\sum_{\xi \in D} u^i(\xi, x(\xi)) < \infty.$$

Therefore there exists $T^0 \geq 1$ such that

$$\sum_{T \geq T^0} \sum_{\xi \in D_T} u^i(\xi, x(\xi)) \leq \frac{\varepsilon}{2}.$$

Since agent i is myopic with respect to default penalties, there exists $T > \max\{t, T^0\}$ such that

$$\sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) H(\xi, j) \leq \frac{\varepsilon}{2}.$$

Consider now the plan \hat{a} defined by

$$\hat{a}(\xi) = \begin{cases} a(\xi) & \text{if } t(\xi) < T \\ (\omega^i(\xi), 0, 0, \hat{d}(\xi)) & \text{if } t(\xi) = T \\ (\omega^i(\xi), 0, 0, 0) & \text{if } t(\xi) > T \end{cases}$$

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Observe that the plan \widehat{a} satisfies

$$\widehat{a} \in B^i(p, q, \kappa) \cap C^T \quad \text{and} \quad \widehat{a}\mathbf{1}_{[0, T-1]} = a\mathbf{1}_{[0, T-1]}.$$

Moreover, for every $\tau \geq T$ we have

$$\begin{aligned} \Pi^{i, T}(p, \widehat{a}) &\geq \Pi^{i, T-1}(p, \widehat{a}) + \sum_{\xi \in D_T} \left[u^i(\xi, \omega^i(\xi)) - \sum_{j \in J} \lambda^i(\xi, j)H(\xi, j) \right] \\ &\geq \Pi^{i, T-1}(p, a) - \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j)H(\xi, j) \\ &\geq \Pi^{i, T-1}(p, a) - \frac{\varepsilon}{2} \\ &\geq \Pi^{i, \tau}(p, a) - \frac{\varepsilon}{2} - \sum_{T \leq s \leq \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \\ &\geq \Pi^{i, \tau}(p, a) - \varepsilon. \end{aligned}$$

It follows that for every $\tau \geq T$

$$\Pi^{i, \tau}(p, \widehat{a}) = \Pi^{i, T}(p, \widehat{a}) + \sum_{\xi \in D^\tau \setminus D^T} u^i(\xi, \omega^i(\xi)) > \Pi^{i, \tau}(p, a) - \varepsilon.$$

Since $T - 1 \geq t$, this implies that the plan a has finite equivalent payoffs. \square

Remark 4.2. Given Proposition 4.1 one may wonder whether restricting plans to have finite equivalent payoffs is relevant to the issue of existence. Since myopia implies that budget and physically feasible plans have finite equivalent payoffs, why one should impose any kind of debt constraints on available plans to ensure existence? The answer to this question lies on the fact that in decentralized economies agents do not take into account feasibility restrictions when they solve their maximization problem. Only budgetary restrictions are relevant for them. But if this is the case, in the absence of borrowing constraints, agents can run a Ponzi scheme and equilibria may fail to exist.²⁰

²⁰Páscoa and Seghir (2009) provide an example of an economy with myopic agents and no borrowing constraints in which equilibria fail to exist.

4.2 Proof of existence

The main contribution of this paper is the following existence result.

Theorem 4.1. *If every agent is myopic with respect to default penalties then a competitive equilibrium with finite equivalent payoffs exists, i.e., $\text{Eq}_\star(\mathcal{E}) \neq \emptyset$.*

We propose a simple proof based on the standard “truncation argument”.

Proof of Theorem 4.1. For each $\tau \in \mathcal{T}$, we denote by \mathcal{E}^τ the truncation of the economy for which the final period is τ . Such economy is a simple extension to multiple periods of the two period model in Dubey, Geanakoplos and Shubik (2005). Adapting their arguments or following the arguments in Páscoa and Seghir (2009),²¹ it is possible to prove that under our set of assumptions, there exist a process of prices and delivery rates $\pi^\tau = (p^\tau, q^\tau, \kappa^\tau)$ and a process of plans $\mathbf{a}^\tau = (a^{i,\tau})_{i \in I}$ such that $(\pi^\tau, \mathbf{a}^\tau)$ is a competitive equilibrium for the truncated economy \mathcal{E}^τ with $\|p^\tau(\xi)\| \geq m(\xi) > 0$ for some $m(\xi)$ that depends only on the primitives of the economy \mathcal{E} (and is independent of the truncation size τ).²²

We denote by $\text{cl}\Xi$ the closure of Ξ under the weak topology.²³ Each process π^τ belongs to $\text{cl}\Xi$ which is weakly compact as a product of compact sets. Passing to a subsequence if necessary, we can assume that the sequence $(\pi^\tau)_{\tau \in \mathcal{T}}$ converges to a process $\pi = (p, q, \kappa)$ in $\text{cl}\Xi$. Observe that, for each node $\xi \in D$, we have $\|p(\xi)\| \geq m(\xi) > 0$. In particular, for each period t and every plan $a \in A$, the payoff $\Pi^{i,t}(p, a)$ is well defined.

By feasibility at each node ξ , we get for each j

$$x^{i,\tau}(\xi) \leq \Omega(\xi), \quad \varphi^{i,\tau}(\xi, j) \leq M(\xi, j) \quad \text{and} \quad \theta^{i,\tau}(\xi, j) \leq M(\xi, j).$$

This implies that the sequence $(x^{i,\tau}(\xi), \varphi^{i,\tau}(\xi), \theta^{i,\tau}(\xi))_{\tau \in \mathcal{T}}$ is uniformly bounded. By optimality, the delivery $d^{i,\tau}(\xi, j)$ is always lower than the promise $V(p^\tau, \xi, j)\varphi^{i,\tau}(\xi, j)$ and therefore the sequence $(d^{i,\tau}(\xi))_{\tau \in \mathcal{T}}$ is uniformly bounded. Passing to a subsequence if necessary, we can assume that for each i , the sequence $(a^{i,\tau})_{\tau \in \mathcal{T}}$ converges to a process $a^i \in A$.

We claim that (π, \mathbf{a}) is a competitive equilibrium with finite equivalent payoffs for the economy \mathcal{E} . It is straightforward to check that each plan a^i belongs to the budget set $B^i(p, q, \kappa)$ and that the feasibility conditions (2.5),

²¹We can also adapt the arguments of the proof of Theorem 1 in Araujo, Páscoa and Torres-Martínez (2002).

²²We refer to the appendix for the precise definition of the truncated economy \mathcal{E}^τ and the associated (finite-horizon) equilibrium concept.

²³The process (p, q, κ) belongs to $\text{cl}\Xi$ if the condition “ $p(\xi) \in \mathbb{R}_{++}^L$ ” in (2.4) is replaced by “ $p(\xi) \in \mathbb{R}_+^L$ ”.

(2.6), (2.7) and (2.8) are satisfied. Applying Proposition 4.1, we get that the plan a^i has finite equivalent payoffs. We propose now to prove that a^i is optimal among plans with finite equivalent payoffs, i.e., the set $\text{Pref}^i(p, a^i) \cap B_\star^i(p, q, \kappa)$ is empty. Assume by way of contradiction that there exist $\varepsilon > 0$, a plan \bar{a} in the budget set $B_\star^i(p, q, \kappa)$ and $T^1 \in \mathbb{N}$ satisfying

$$\forall T \geq T^1, \quad \Pi^{i,T}(p, \bar{a}) > \Pi^{i,T}(p, a^i) + \varepsilon. \quad (4.1)$$

Since a^i is physically feasible, we have $x^i(\xi) \leq \Omega(\xi)$ for every node $\xi \in D$. It follows from Assumptions (H.2) and (H.3) that $U^i(x^i) \leq U^i(\Omega) < +\infty$. This implies that the limit

$$\Pi^i(p, a^i) \equiv \lim_{T \rightarrow \infty} \Pi^{i,T}(p, a^i)$$

exists in $[-\infty, \infty)$. In particular, there exists $T^2 \geq T^1$ such that

$$\forall T \geq T^2, \quad \Pi^{i,T}(p, a^i) + \frac{\varepsilon}{2} > \Pi^i(p, a^i). \quad (4.2)$$

Since the plan \bar{a} has finite equivalent payoffs, there exists $T > T^2$ and \tilde{a} in the set $B^i(p, q, \kappa) \cap C^T$ such that

$$\tilde{a}\mathbf{1}_{[0, T^2]} = \bar{a}\mathbf{1}_{[0, T^2]} \quad \text{and} \quad \inf_{\tau \geq T} [\Pi^{i,\tau}(p, \tilde{a}) - \Pi^{i,\tau}(p, \bar{a})] \geq -\frac{\varepsilon}{4}.$$

We denote by \hat{a} the plan defined by

$$\forall \xi \in D, \quad \hat{a}(\xi) = \begin{cases} \tilde{a}(\xi) & \text{if } t(\xi) \leq T \\ (0, 0, 0, 0) & \text{if } t(\xi) > T. \end{cases}$$

Observe that \hat{a} belongs to the truncated budget set $B^i(p, q, \kappa) \cap B^T$ and satisfies

$$\hat{a}\mathbf{1}_{[0, T^2]} = \bar{a}\mathbf{1}_{[0, T^2]} \quad \text{and} \quad \Pi^{i,T}(p, \hat{a}) \geq \Pi^{i,T}(p, \bar{a}) - \frac{\varepsilon}{4}. \quad (4.3)$$

Combining (4.1), (4.2) and (4.3) we get

$$\Pi^{i,T}(p, \hat{a}) > \Pi^i(p, a^i) + \frac{\varepsilon}{4}.$$

We let ψ^i be the correspondence from A to A^T defined by

$$\forall a \in A, \quad \psi^i(a) = \left\{ b \in B^T : \Pi^{i,T}(p, b) > \Pi^i(p, a) + \frac{\varepsilon}{4} \right\}.$$

Let F^i be the correspondence from $\Xi \times A$ to A^T defined by

$$\forall (\pi', a') \in \Xi \times A, \quad F^i(\pi', a') = B^{i,T}(\pi') \cap \psi^i(a').$$

Observe that $\hat{a} \in F^i(\pi, a^i)$. Moreover, we proved that there exists a strictly increasing sequence $(T_n)_{n \in \mathbb{N}}$ with $T_n \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} ((p_n, q_n, \kappa_n), a_n^i) = ((p, q, \kappa), a^i)$$

where

$$(p_n, q_n, \kappa_n) = (p^{T_n}, q^{T_n}, \kappa^{T_n}) \quad \text{and} \quad a_n^i = a^{i, T_n}.$$

Since F^i is lower semi-continuous on $\Xi \times A$ for product topologies,²⁴ we can conclude that there exists $\nu \in \mathbb{N}$ large enough such that $T_\nu \geq T$ and the set $F^i((p_\nu, q_\nu, \kappa_\nu), a_\nu^i)$ is non-empty. Let \hat{a}_ν be an element of that set. This means that

$$\hat{a}_\nu \in B^{i, T}(p_\nu, q_\nu, \kappa_\nu) \quad \text{and} \quad \Pi^{i, T}(p_\nu, \hat{a}_\nu) \geq \Pi^i(p_\nu, a_\nu^i) + \frac{\varepsilon}{4}.$$

Since $T_\nu \geq T$, we have

$$B^{i, T}(p_\nu, q_\nu, \kappa_\nu) \subset B^{i, T_\nu}(p_\nu, q_\nu, \kappa_\nu) \quad \text{and} \quad \Pi^{i, T_\nu}(p_\nu, \hat{a}_\nu) \geq \Pi^{i, T}(p_\nu, \hat{a}_\nu).$$

It follows that

$$\Pi^{i, T_\nu}(p_\nu, \hat{a}_\nu) > \Pi^i(p_\nu, a_\nu^i) = \Pi^{i, T_\nu}(p_\nu, a_\nu^i)$$

contradicting the optimality of a_ν^i in the truncated economy \mathcal{E}^{T_ν} under the price process $(p_\nu, q_\nu, \kappa_\nu)$.

We have thus proved that for each i , the plan a^i has finite equivalent payoffs and satisfies

$$\text{Pref}^i(p, a^i) \cap B_\star^i(p, q, \kappa) = \emptyset.$$

This means that a^i belongs to the demand set $d_\star^i(\pi)$. We already proved that all markets clear. This means that (π, \mathbf{a}) is a competitive equilibrium with finite equivalent payoffs. \square

Given Proposition 3.3, we can obtain the main existence result in Araujo, Páscoa and Torres-Martínez (2002, Theorem 2) as a direct corollary of Theorem 4.1.

Corollary 4.1 (Araujo, Páscoa and Torres-Martínez (2002)). *If there is no default penalty then there exists a competitive equilibrium, i.e., $\text{Eq}(\mathcal{E}) \neq \emptyset$.*

²⁴See Páscoa and Seghir (2009) for detailed arguments.

4.3 General default penalties

Our aim is to illustrate that the borrowing constraints introduced by Levine and Zame (1996), when suitably modified, are particularly well-adapted to preclude Ponzi schemes in models with limited commitment and collateral requirements. To illustrate this, we have chosen a familiar and simple framework, modeling agents' payoffs as in Dubey, Geanakoplos and Shubik (1990) (see also among others Zame (1993), Dubey, Geanakoplos and Shubik (2005) and Páscoa and Seghir (2009)). That is, each agent's payoff equals the utility for consumption minus the default penalty assumed to be a linear function of the amount of default expressed in real terms. We could alternatively consider a more general framework where the payoff $\Pi^{i,t}(p, a)$ is described as follows

$$\Pi^{i,t}(p, a) \equiv \sum_{\xi \in D^t} \Pi^i(\xi, x(\xi), \delta(p(\xi), \varphi(\xi^-), d(\xi)))$$

where

$$\delta(p(\xi), \varphi(\xi^-), d(\xi)) = (\delta_j(p(\xi), \varphi(\xi^-), d(\xi)))_{j \in J}$$

with

$$\delta_j(p(\xi), \varphi(\xi^-), d(\xi)) \equiv \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}$$

and $\Pi^i(\xi, \cdot, \cdot)$ is a function from $\mathbb{R}_+^L \times \mathbb{R}_+^J$ to $[-\infty, \infty)$. The number $\Pi^i(\xi, x, \delta)$ represents the payoff received (or felt) by agent i at $t = 0$ if at node ξ , he consumes the bundle $x \in \mathbb{R}_+^L$ and defaults on each asset j the quantity $\delta_j \geq 0$ expressed in real terms. Observe that the model considered in this paper is a particular case where

$$\Pi^i(\xi, x, \delta) = u^i(\xi, x) - \sum_{j \in J} \lambda^i(\xi, j)\delta_j.$$

We can replace Assumptions H.2 and H.3 by the conditions

(G.2) for every node ξ , the function $(x, \delta) \mapsto \Pi^i(\xi, x, \delta)$ is concave, continuous, strictly increasing on x , decreasing on δ with $\Pi^i(\xi, 0, 0) = 0$;

(G.3) the infinite sum $\sum_{\xi \in D} \Pi^i(\xi, \Omega(\xi), 0)$ is finite.

If we replace the condition that each agent i is myopic with respect to default penalties by the following one

$$\liminf_{T \rightarrow \infty} \sum_{\xi \in D_T} \Pi^i(\xi, 0, H(\xi)) = 0$$

then we can reproduce all the arguments of the paper.

Imposing utility penalties as a punishment for default is obviously a specific modeling choice. In the two-period model of Dubey, Geanakoplos and Shubik (2005) (see also the two-period model of Zame (1993)) one could argue that this serves as a short-cut because one cannot model (temporary or definitive) exclusion from financial markets and reputational effects of default. One may argue that the main advantage of using an infinite horizon dynamic model is the possibility of modeling explicitly pecuniary punishment, like garnishment of future income or denial of credit in all future periods.²⁵ We share this opinion, however, the treatment of pecuniary punishments involves serious technical complications (non-convexities) that we leave for future research.

5 Conclusion

What makes general equilibrium models with collateral requirements (Araujo, Páscoa and Torres-Martínez (2002) and Kubler and Schmedders (2003)) very appealing is that collateral constraints not only do exist in actual markets but seem to be an efficient mechanism to preclude Ponzi schemes without imposing any *ad-hoc* constraint on debt. The recent contributions of Páscoa and Seghir (2009) and Ferreira and Torres-Martínez (2010) show that the positive results in Araujo, Páscoa and Torres-Martínez (2002) may not be robust: the effectiveness of collateral requirements to bound debt may not be valid anymore in the natural case where there are other mechanisms leading agents to overpay, that is, to repay more than the collateral when the value of their debt actually exceeds the collateral value.

To formally close the model and restore equilibrium, we need to impose borrowing constraints. Among the different approaches already existing in the literature with full commitment, we argue in favor of the endogenous debt constraints à la Levine and Zame (1996). Assuming that each agent conceives that at any time period he may be restricted to have access to borrowing for a finite number of periods, we introduce in the setting of Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of plans with *finite equivalent payoffs*. When payments are fully enforced, our concept of plans with finite equivalent payoffs coincides with the concept of plans with finitely effective debt introduced by Levine and Zame (1996). When there are collateral requirements but no default penalties, any budget feasible plan has automatically finite equivalent payoffs. In particular, our budget set coincides with the standard one defined in Araujo, Páscoa and

²⁵Non-pecuniary punishment like prison terms or pangs of conscience must be treated through penalties directly assessed in terms of utility.

Torres-Martínez (2002) and Kubler and Schmedders (2003). Assuming a mild assumption on default penalties, namely that agents are myopic with respect to default penalties, we show that restricting actions to have finite equivalent payoffs rules out Ponzi schemes and guarantees equilibrium existence while keeping the minimal ability to borrow and lend that we expect in our model. The proof is very simple and intuitive. In particular, the main existence result in Araujo, Páscoa and Torres-Martínez (2002) is a direct corollary of our existence result.

Appendix

Fix $\tau \in \mathcal{T}$ with $\tau > 0$. Recall that A^τ denotes the set of all plans $a \in A$ such that

$$\forall \xi \in D, \quad t(\xi) > \tau \implies a(\xi) = 0.$$

We let B^τ be the set of plans $a \in A^\tau$ satisfying the additional condition

$$\forall \xi \in D, \quad t(\xi) = \tau \implies \varphi(\xi) = 0.$$

Given a process $(p, q, \kappa) \in \Xi$, we denote by $B^{i,\tau}(p, q, \kappa)$ the set defined by

$$B^{i,\tau}(p, q, \kappa) \equiv B^i(p, q, \kappa) \cap B^\tau.$$

A competitive equilibrium for the *truncated* economy \mathcal{E}^τ is a family of prices and delivery rates $\pi = (p, q, \kappa) \in \Xi$ and an allocation $\mathbf{a} = (a^i)_{i \in I}$ with $a^i \in B^\tau$ such that

(a) for every agent i , the plan a^i is optimal, i.e.,

$$a^i \in d^{i,\tau}(p, q, \kappa) \equiv \operatorname{argmax}\{\Pi^{i,\tau}(p, a) : a \in B^{i,\tau}(p, q, \kappa)\}; \quad (.1)$$

(b) commodity markets clear at every node up to period τ , i.e.,

$$\sum_{i \in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \quad (.2)$$

and for all $\xi \in D^\tau \setminus \{\xi_0\}$,

$$\sum_{i \in I} x^i(\xi) = \sum_{i \in I} [\omega^i(\xi) + Y(\xi)x^i(\xi^-)]; \quad (.3)$$

(c) asset markets clear at every node up to period $\tau - 1$, i.e., for all $\xi \in D^{\tau-1}$,

$$\sum_{i \in I} \theta^i(\xi) = \sum_{i \in I} \varphi^i(\xi); \quad (.4)$$

(d) deliveries match up to period τ , i.e., for all $\xi \in D^\tau \setminus \{\xi_0\}$ and all $j \in J$,

$$\sum_{i \in I} V(\kappa, p, \xi, j) \theta^i(\xi^-, j) = \sum_{i \in I} d^i(\xi, j). \quad (.5)$$

Observe that if a plan a belongs to B^τ , then $\Pi^{i,\tau}(p, a)$ and $\Pi^i(p, a)$ coincide for every price process p . Moreover, if (π, \mathbf{a}) is a competitive equilibrium for the truncated economy \mathcal{E}^τ , then without any loss of generality, we can assume that $q(\xi) = 0$ and $\theta(\xi) = 0$ for every terminal node $\xi \in D_\tau$.

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