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# Discrete Public Goods with Incomplete Information†

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**Abstract**

We analyze simultaneous discrete public good games with incomplete information and continuous contributions. To use the terminology of Admati and Perry (1991), we consider contribution and subscription games. In the former, contributions are not refunded if the project is not completed, while in the latter they are. For the special case where provision by a single player is possible we show the existence of an equilibrium in both contribution and subscription games where a player decides to provide the good by himself. For the case where is not feasible for a single player to provide the good by himself, we show that any equilibrium of both games is inefficient. WE also provide a sufficient condition for "contributing zero" to be the unique equilibrium of the contribution game with $n$ players and characterize equilibria of the subscription game involving positive contributions.

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# 1 Introduction

The literature on private provision of public goods can be divided into two broad categories.[1] The first branch of the literature focuses on the provision of continuous public goods. Papers include Warr (1982, 1983), Cornes and Sandler (1984), Bergstrom, Blume and Varian (1986), Andreoni (1988), Gradstein, Nitzan and Slutsky (1994) and others. A standard result in this literature is that public goods are underprovided by voluntary contributions due to free riding behavior.[2] One might conjecture that the government can solve this underprovision problem by providing some of the good and financing it by imposing taxes on contributors. However, Warr (1982) and Roberts (1984), in two influential papers, show that government contributions result in a dollar to dollar reduction in private contributions if the tax on contributors does not change the set of contributing individuals. Bergstrom, Blume and Varian (1986) show that the crowding out effect is only partial if one allows for the taxes that pay for the government contribution to be also collected from non contributors. Hence the literature suggests that underprovision is a robust conclusion if the public good level is endogenously determined by voluntary contributions.[3]

[1] In this paper, we focus on private provision. There is an extensive mechanism design literature that treats public provision. Papers in this literature develop efficient mechanisms for the provision of public goods. These mechanisms are in general complex and require a central authority to implement them and hence would be best decsribed as mechanisms for public provision of public goods. Gradstein (1994) examines efficient mechanisms for discrete public goods. Other papers that treat public provision include Maskin (1977), d'Aspremont and Varet (1979, 1982), Palfrey and Srivastava (1986). Cornelli (1996) examines an optimal mechanism for a monopolist which produces an excludable good that has large fixed costs.

[2] The free riding problem becomes worse in the presence of incomplete information. Gradstein (1992) considers a dynamic model of private provision for a continuous public good with incomplete information with the restriction that players either contribute zero or an exogenous positive amount. He concludes that in addition to the standard underprovision results, inefficiency occurs because of a delay in contributions. This inefficiency does not disappear as the population becomes large.

[3] Andreoni (1998) analyzes the role of seed money in the presence of nonconvexities in the production of the public good. He shows that a small amount of seed money can

The second strand of the literature focuses on discrete public goods where a fixed level of a public good is provided if enough contributions are collected to cover its cost $c$. Otherwise, the good is not provided. Typical examples include building a bridge, a library of a certain size, public radio fund raising to finance a certain program. In all of these examples, if enough money is raised to cover the cost of the public good, then the good is provided, otherwise the good is not provided.

Palfrey and Rosenthal (1984) developed the first modern treatment of private provision of discrete public goods. More specifically they analyze contribution and subscription games — to use the terminology of Admati and Perry (1991) — for a discrete public good under complete information where players' strategies are restricted to either contribute zero or an exogenous positive amount. In a contribution game, contributions are not refunded if the sum of the contributions does not cover the cost of the public good $c$, while in a subscription game players get their money back if the project is not completed. An example with the main features of a subscription game can be given as follows. The Wisconsin Governor has recently pledged $27 million in state bonds to finance a new $72 million basketball arena on the condition that the rest of the money be raised by private donations (Andreoni, 1998). That is, the Governor will provide $27 million as long as the remaining $45 million is raised, otherwise the offer is cancelled. Other examples can be found in the fund-raising literature. Examples of contribution games include public radio and TV fund-raising efforts where contributors do not get their money back if the program is not provided. Other examples of contributions games are situations where contributions take the form of physical labor, in this case volunteers cannot recover their effort if the project is not completed.

In contrast with the standard underprovision result for continuous public goods, Palfrey and Rosenthal (1984) obtain the startling conclusion that efficient provision of a discrete public good is a *possible* outcome of both contribution and subscription games even though inefficient equilibria also exist. Bagnoli and Lipman (1989) extend Palfrey and Rosenthal model by allowing individuals to make continuous contributions. They show that the set of undominated perfect equilibrium outcomes of the subscription game is not only efficient but coincides with the core of the economy. They also show that with a dynamic version of the subscription game, it is possible to obtain efficient outcomes even if the level of the public good is not binary

generate a substantial amount of voluntary contributions.

as long as the number of units of the public good is countable.[4] It follows that the general conclusion from the theoretical literature is that private provision of discrete public goods with complete information *can* be efficient. Moreover, efficient provision is a possible outcome for both subscription and contribution games.

An important question that has not been addressed in the literature on voluntary provision of discrete public goods is to what extent these results generalize to a model where players have incomplete information about other players' valuation of the public good. Moreover, there is both casual evidence from the fund-raising literature of the superiority of subscription games and experimental evidence from Bagnoli and McKee (1991) and Cadsby and Maynes (1999) that a refund increases the chance of providing the good. More specifically, Cadsby and Maynes consider a discrete public good experiment. They provide experimental evidence showing that provision is encouraged in a subscription game vis-a-vis a contribution game. They also provide evidence that a high $c$ discourages provision in the contribution game but not in the subscription game. Our results will confirm these findings.

In this paper, we examine contribution games and subscription games for a discrete public good in the presence of incomplete information. We first analyze provision games where it is feasible for a player to provide the public good by himself. Common examples include opening a window in a hot room, rescuing an injured person in a traffic accident, etc. We show that when the cost of provision $c$ is not prohibitively high as to prevent a single player from providing the good, there always exists an equilibrium where a player with a sufficiently large valuation provides the good in both the subscription and the contribution game. Moreover, we show that the free riding problem can become more severe or less severe as the population becomes larger.

We also examine contribution and subscription games when contributions

[4] Admati and Perry analyzed the role of commitment in the provision of public goods. More specifically, they examined two-player contribution and subscription games with complete and perfect information where players alternate in making pledges or contributions until the public project is completed. They show that with linear costs, a necessary and sufficient condition for completion of the project under a contribution game is that each player would complete the project immediately by himself if he was the only player. We provide a sufficient condition and with two or more players for "contributing zero" to be the unique equilibrium of the contribution game. For the subscription game, they show that all socially desirable projects are completed in equilibrium. In contrast, we show that with incomplete information the probability of provision conditional on being efficient to do so is always less than one.

or pledges from more than one individual are necessary in order for the good to be provided. We will characterize some symmetric equilibrium of the subscription game and demonstrate that all equilibria are inefficient, that is, the probability of provision given that it is efficient to do so is less than one. This inefficiency stems from the difficulties arising in coordinating to overcome the free-rider problem in the presence of incomplete information. We will also provide a sufficient condition for 'contributing zero' to be the unique equilibrium of the contribution game. Therefore, we show that under some circumstances *the contribution game is never more efficient than the subscription game*. This confirms the evidence from the experimental literature of the superiority of subscription games over contribution games. In contrast to the mechanism design literature, our approach of characterizing equilibrium behavior in existing mechanisms enables us to provide results which can be tested in a laboratory environment or by using empirical data.

This remainder of the paper provides more details on all of these issues. In Section 2, the specifics of the model are presented. Section 3 provides an example that illustrates the main results of the paper. In Section 4, the case where a single individual can provide the good by himself is analyzed. The general analysis of subscription games is contained in Section 5. Section 6 examines the contribution game. Section 7 illustrates a differential equations approach to find equilibrium strategies in games with two players. Our main results are summarized in Section 8.

# 2 The Model

We now present the formal model. There are $N \geq 2$ individuals with private independent values for a discrete public good. That is, each individual $i$, $i = 1, ..., N$ knows his own value $v_i$ but only the distribution of his opponents' values. Values are assumed to be independently drawn from a continuous distribution $F$. The cost of providing the public good is $c$.

With the exception of Section 4, we assume throughout the paper that an individual cannot provided the good by himself. In this case, $F(c) = 1$ and thus the distribution has a bounded support. As a normalization we suppose, without loss of generality, the support to be contained in the interval $[0, 1]$.

We will examine both *subscription and contribution* games. In a subscription game the money contributed by players is returned if the sum of the pledges is less than $c$. In a contribution game the money is not returned even if the good is not provided. Whatever the method to elicit donations is, if donations add up to more than $c$, the additional money is not returned to the contributors.

More specifically, we analyze one-shot simultaneous move contribution and subscription games for a discrete public good in the presence of incomplete information about preferences.[5] Our model also relaxes the binary contribution restriction imposed in the literature.[6] While there are important instances where binary contributions are relevant due to transaction costs, in general players can give any amount of money they desire (alumni donation, donations to a library, etc.). Moreover, in a continuous contributions framework, individuals make contributions that best match their preferences as opposed to a discrete contribution model. Cadsby and Maynes (1999) provide experimental evidence showing that allowing continuous rather than binary "all-or-nothing" contributions facilitates provision.

As it is standard in many games of incomplete information with ex-ante symmetric players — such as in industrial organization and auctions — our focus is mainly on symmetric equilibrium. Nevertheless, we provide inefficiency results that hold for both symmetric and asymmetric equilibria; we show that for *any equilibrium* the probability of provision of the public good given that it is efficient to do so is less than one in both contribution and subscription games and it is frequently equal to zero in contribution games. We also provide sufficient conditions under which contributing zero is the *unique* equilibrium of the contribution game.

The multiplicity of equilibria that characterizes these games has two sources. Firstly, this is an instance of a coordination game. With two players who cannot provide the good by themselves, the coordination nature of the game is clear. With more than two players, then we have to account for all possible combinations of players who can provide the good as a group. This adds a new layer of complexity to standard coordination games. Secondly,

[5]It is important to note that there are several papers that introduce incomplete information in one form or another but do not address the above questions. Palfrey and Rosenthal (1988) analyze the provision of a discrete public good when individuals have incomplete information about the degree of altruism of other players under the restriction that players are only allowed to make discrete contributions. Nitzan and Romano (1990) show that when the cost of the discrete public good is uncertain to the players, then efficiency is no longer obtained.

[6]Bagnoli and Lipman (1989) also considers continuous contributions. However their model is with complete information.

the incompleteness of information and the assumption that player can contribute or pledge any values they like also creates additional coordination issues beyond the standard coordination problem. It is the combination of these two features that makes it difficult to obtain uniqueness of equilibrium. In particular, as we will explain in the context of the example below, well-known arguments to obtain uniqueness, such as the iterated deletion of dominated strategies as in Bagnoli and Lipman (1989) (with complete information) and Milgrom and Roberts (1990) (with incomplete information) for the case of supermodular games, do not apply in this context.

# 3 An Example

Before turning to an analysis of the model, it is instructive to consider a simple example where it is possible to make the point that the efficient results in the discrete public goods literature are not robust to the introduction of incomplete information. This example incorporates the simplest structure for which this lack of robustness is evident and for which we obtain a striking distinction between predicted outcomes in subscription and contribution games. We begin with the complete information case.

## 3.1 Complete Information

Assume that players 1 and 2 have valuations for the public good $v_1$ and $v_2$ in [0,1] and that are common knowledge. Let the cost of provision of the public good $c$ be such that $1 < c < 2$ and $v_1 + v_2 \geq c$. That is, a single player cannot provide the good by himself and it is socially efficient to provide the good. If we denote by $b_i, i = 1, 2$, player $i$'s pledging strategy in a subscription game, then we can characterize the set of (pure strategy) equilibria as follows:

**(i)** $\{(b_1, b_2) : b_1 + b_2 = c; b_1 \leq v_1; b_2 \leq v_2\}$;

**(ii)** $(0, 0)$;

**(iii)** $\{(b_1, b_2) : b_1 < c - v_2; b_2 < c - v_1\}$.

The equilibria described in (i) are efficient, while the equilibria described in (ii) and (iii) are not.[7] Moreover, the equilibria described in (ii) and (iii)

[7] The equilibrium described in (ii) is known as the strong free riding equilibrium.

are not strict as best responses are not unique.[8]

Note that the equilibria of the contribution game can be described by (i) and (ii) above. Given that individuals in a contribution game never receive their contributions back, the strategies profiles described in (iii) are no longer equilibria. Unlike in the case of the subscription game, the equilibrium strategy $(0,0)$ is now a strict equilibrium. That is, contributing zero is the unique best response when the other player contributes zero.

This example suggests that under complete information the distinction between contribution and subscription game both in terms of efficiency and in terms of expected outcomes is unclear. We now consider the incomplete information case.

## 3.2 Incomplete Information

We now assume that $v_1$ and $v_2$ are independent draws from the uniform [0,1] distribution. Moreover, Player $i = 1, 2$ knows his own valuation but only the distribution of his opponent's valuation. Let us suppose that $1 < c < 2$. The subscription game has many symmetric equilibria and the contribution game has only the strong free riding equilibria. Let us see some subscription game equilibria first:

**i.** $(0,0)$;

**ii.** $b^*(v) = \begin{cases} 0, \text{ if } v < \frac{c}{2} \\ \frac{c}{2}, \text{ if } v \geq \frac{c}{2} \end{cases}$;

**iii.** $b^*(v) = \begin{cases} \frac{2c-1}{6} + \frac{v}{2}, & \text{if } \frac{2c-1}{3} \leq v \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

That $(0,0)$ is still an equilibrium in the incomplete information follows trivially from the fact that a single player cannot provide the good by himself. Now let's check that the strategy profile described in (ii) is indeed an equilibrium. Suppose Player 2 follows $b^*(\cdot)$. If Player 1's value is less than $\frac{c}{2}$, then 1's best reply is to pledge any number less than $\frac{c}{2}$. In particular, pledging zero is a best response. If Player 1's value is greater than equal to $\frac{c}{2}$, then Player 1's best response is to pledge $\frac{c}{2}$. Pledging less than $\frac{c}{2}$ Player

[8] Bagnoli and Lipman (1989) show that the inefficient equilibria in (ii) and (iii) can be eliminated by undominated perfection.

1 obtains zero profits; by pledging more than $\frac{c}{2}$, Player 1 does not increase the probability that the good is provided but decreases profits conditional on provision.

The equilibrium pledging strategies described in (iii) can be explained as follows. A player pledges the equivalent to the expected value of the other player being lower than his own, conditional on the interval $[\frac{2c-1}{3}, 1]$, that is, on the relevant interval where pledges are less than or equal to the valuations for the public good, i.e. $b(v) \leq v$. A formal derivation is left to the appendix.

Note that the iterated elimination of dominated strategies does not bite here. In our setting, in a first round of elimination Player $i$ eliminates strategies $b_i(v_i) > v_i$. The elimination process stops there! The type of bounds obtained in Milgrom and Roberts(1990) are not possible here since the game is not supermodular: Consider two strategies $b_1 = 0$ and $b_2 = c$ and note that $f(b_2, b_2) - f(b_2, b_1) - [f(b_1, b_2) - f(b_1, b_2)] = (v - c) - (v - 0) = -v < 0$.

Of course this multiplicity of equilibria can potentially raise difficulties in questions of welfare. Nevertheless, we will show later that *any equilibria* of the subscription game is inefficient. More precisely we will show that the probability of provision of the public good given that it is efficient to provide is (uniformly)less than one in any equilibrium.[9] Moreover, we will provide a sufficient condition under which the welfare comparison between subscription and contribution games is straightforward — as the only equilibrium of the contribution game is to contribute zero.

We now return to the example and analyze the contribution game. This example satisfies this sufficient condition that will be specified later. However, it may be instructive to show directly that (0,0) is the unique equilibrium. Suppose $(b_1, b_2)$ is an equilibrium of the contribution game. It is immediate that $b_2(v_2) \leq v_2$. Let us find the best response of Player 1 to $b_2$. If Player 1 bids $x \geq 0$ his expected utility is

$$\phi(x) = vP(b_2(v_2) \geq c - x) - x.$$

If $x \neq 0$ and $c - x \geq 1$ then $\phi(x) = -x < 0$. Now suppose $c - x < 1$. We have

$$\phi(x) \leq vP(v_2 \geq c - x) - x = v(1 - (c - x)) - x = v(1 - c) + (v - 1)x < 0.$$

[9]This probability is equal to 1/2 in the equilibrium described in (ii) and equal to 2/3 in the equilibrium described in (iii).

Thus $x = 0$ is the best response. (Note that we are *not* limiting ourselves to symmetric pure strategy equilibrium). Therefore, despite the multiplicity of equilibria of the subscription game, we are still able to make welfare comparisons. More precisely, the contribution game is less efficient than the subscription game in this example. This result contrasts greatly with the case where players have complete information and both games have efficient equilibria.

# 4 Provision by a Single Player

In this section, we analyze contribution and subscription games when valuations for the public good can take values larger than the cost. The following theorem shows that there always exists a symmetric equilibrium where a player with a sufficiently high valuation provides the good by himself.

**Theorem 1** *Suppose $F : [0, \infty) \to \mathbb{R}$ is a continuous distribution. Suppose there are $N \geq 2$ players for a project with cost $c > 0$ and that $F(c) < 1$. Then there exists an $\alpha > 0$ where $\alpha$ solves $\alpha F(\alpha)^{N-1} = c$ such that*

$$b(v) = \begin{cases} 0 & \text{if} \quad v \leq \alpha \\ c & \text{if} \quad v > \alpha \end{cases}$$

*is an equilibrium strategy for both contribution and subscription games.*

**Proof.** Suppose players $n = 2, ..., N$ play according to $b(\cdot)$. Let us find the best response of Player 1. If his value is $v \geq 0$ and his contribution is $x \geq 0$, his expected payoff in the contribution game is given by

$$g(x) = v \Pr(x + b(v_2) + ... + b(v_N) \geq c) - x$$

Since $b(v_2) + ... + b(v_N)$ is either 0 or not less than $c$, it follows that if $x < c$ then $g(x) \leq g(0)$. Thus $\max\{g(x); x \geq 0\} = \max\{g(0), g(c)\} = \max\left\{v\left(1 - F(\alpha)^{N-1}\right), v - c\right\}$. Hence if $v < \frac{c}{F(\alpha)^{N-1}} = \alpha$, the best contribution is $x = 0$. If $v > \alpha$ the best contribution is $x = c$. And if $v = \alpha$ the player is indifferent between $x = 0$ and $x = c$. For the subscription game, the expected surplus is given by

$$f(x) = (v - x) \Pr(x + b(v_2) + ... + b(v_N) \geq c)$$

The proof is identical to the contribution game since $f(0) = g(0)$ and $f(c) = g(c)$.[10] ■

**Remark 1** *This equilibrium is not ex-post efficient. Although efficiency requires that the public good be provided whenever the sum of individual's valuations exceed c, the good is provided only if at least one individual's valuation exceeds $\alpha$ which is greater than c. Moreover we will show later that any equilibrium is inefficient.*

Theorem 1 implies that when the cost of provision is not
prohibitively high as to prevent a single player from providing the good, there always exists an equilibrium where a player with a *sufficiently large valuation* provides the good by himself.

Bliss and Nalebuff (1984) analyze a war of attrition game where the public good is provided by a single individual. They show that in this dynamic framework the free rider problem is less important as the population becomes larger and provide conditions under which the first best is achieved as the population approaches infinity. Hence, a natural question is whether in our framework, the probability that the good gets provided increases or decreases as the population becomes larger. The answer to this question depends on the elasticity of the distribution function. To see this, first note that $\alpha_N$ is increasing in $N$ and that the probability that the good is provided is

$$1 - F(\alpha_N)^N = 1 - \frac{F(\alpha_N)}{\alpha_N} c.$$

Taking the derivative of this with respect to $\alpha_N$, we conclude that the derivative is positive if $\frac{F'(\alpha_N)\alpha_N}{F(\alpha_N)} < 1$ and negative if this term is greater than one. That is, free riding can become less severe or more severe as the population becomes larger. There is no presumption that it will get worse or that it will get better. It depends on the distribution function. For example, if $F$ is uniform, then the elasticity is equal to one and the probability that the good gets provided does not change as $N$ increases.

[10]The solution is not unique in general. For example if $N = 2, F(x) = x, x \in [0, 1]$, then if $0 < c < 1/e$ another equilibrium strategy for the contribution game is $b(x) = \max\{c + k \log(x), 0\}$ where $k$ is such that $k^k = e^{-c}$. There are two solutions: $k_1 < 1/e < k_2$.

# 5 The Subscription Game

In this section we analyze the subscription game where a single player cannot provide the good by himself. This is the case only if $F(c) = 1$. Thus the distribution $F$ has a bounded support and without loss of generality we suppose the support is the interval $[0, 1]$. Players' values are determined by independent draws from a continuous distribution $F : [0, 1] \to \mathbb{R}$. The cost of the public good is $c \in [m, m + 1)$ where $m \geq 1$ i.e. $m + 1$ is the minimum number of players needed to provide the public good.

If players $i = 2, 3, \ldots, N$ follow the function $b(\cdot)$ and player 1 with valuation $v$ contributes $x \geq 0$, his expected surplus is

$$\begin{aligned} \phi(x) &= (v - x) \Pr(x + b(v_2) + \ldots + b(v_N) \geq c) \\ &= (v - x) \Pr(b(v_2) + \ldots + b(v_N) \geq c - x) \end{aligned} \quad (1)$$

We see from the above that if $N > 2$, to write $\phi(x)$ as a function of the distribution of individual valuations, $F$, detailed information on $b(\cdot)$ is needed; it does not suffice to know that $b(\cdot)$ is increasing in the private valuation. Hence the standard technique that is used to characterize equilibrium involving increasing strategies in games of incomplete information cannot be used in this problem when $N > 2$. Moreover, it would be fruitless to try to guess all the functional forms that equilibria might take. Thus, it would be extremely difficult to find all the equilibria of the subscription game.

In this section, we show directly that there exists an equilibrium with cut-off strategies where the good is provided if $m + 1$ players have a value greater than a certain cut-off value. We then show that for any equilibrium, the probability that the good gets provided given that it is efficient to so is less than one.

**Proposition 2** *Suppose that players's valuations are distributed in the* $[0, 1]$ *interval. Suppose the cost of the public good is* $c \in [m, m+1)$ *and the number of players* $N \geq m + 1$*. The following strategy is a symmetric equilibrium strategy of the subscription game:*

$$b(v) = \begin{cases} 0 & \text{if} \quad v \leq a; \\ \frac{c}{m+1} & \text{if} \quad a \leq v \leq 1. \end{cases} \quad (2)$$

*where* $a$ *is such that*

$$a \frac{C_{N-1}^{m} (1 - F(a))^{m} F(a)^{N-1-m}}{\sum_{h=m}^{N-1} C_{N-1}^{h} (1 - F(a))^{h} F(a)^{N-1-h}} = \frac{c}{m+1}. \quad (3)$$

*and* $C_n^h = \frac{n!}{h!(n-h)!}$.

**Proof.** We first prove the existence of $a$. Define

$$\begin{aligned} h(a) &= a\frac{C_{N-1}^m (1-F(a))^m F(a)^{N-1-m}}{\sum_{h=m}^{N-1} C_{N-1}^h (1-F(a))^h F(a)^{N-1-h}} \\ &= \frac{aC_{N-1}^m}{C_{N-1}^m + \sum_{h=m+1}^{N-1} C_{N-1}^h (1-F(a))^{h-m} F(a)^{m-h}}. \end{aligned}$$

Note that

$$\lim_{a\to 0} h(a) = 0 \text{ and } \lim_{a\to 1} h(a) = \frac{C_{N-1}^m}{C_{N-1}^m} = 1$$

Since $h$ is a continuous function, by the intermediate value theorem there exists an $a$ such that $h(a) = \frac{c}{m+1} \in (0,1)$.

We now prove that $b(\cdot)$ is an equilibrium. Suppose $v < 1$ and player 1 pledges $x \geq 0$ and players $n = 2, \ldots, N$ follow $b(v_n)$. The expected utility of player 1 is given by

$$\phi(x) = (v - x)\Pr\left(\sum_{n=2}^{N} b(v_n) \geq c - x\right).$$

The range of $\sum_{n=2}^N b(v_n)$ is $\left\{0, \frac{c}{m+1}, \frac{2c}{m+1}, \ldots, \frac{(N-1)c}{m+1}\right\}$.

If $c - x \in (\frac{jc}{m+1}, \frac{(j+1)c}{m+1}]$, then

$$\phi(x) = (v - x)\Pr\left(\sum_{n=2}^{N} b(v_n) \geq \frac{(j+1)c}{m+1}\right) \leq \phi\left(c - \frac{(j+1)c}{m+1}\right)$$

Thus the optimal pledge $x^* \leq v$ is such that

$$x^* \in \left\{c - \frac{(j+1)c}{m+1};\ j \geq -1\right\}.$$

Thus

$$x^* = c - \frac{(j^*+1)c}{m+1} \in [0,1),\ j^* \geq -1.$$

Therefore $j^* \in \{m, m-1\}$ since $\frac{2c}{m+1} \geq 2\frac{m}{m+1} \geq 1$. Finally we have $j^* = m-1$ if and only if $\phi\left(\frac{c}{m+1}\right) \geq \phi(0)$ or equivalently if and only if

$$\left(v - \frac{c}{m+1}\right) \Pr\left(\sum_{n=2}^{N} b(v_n) \geq c - \frac{c}{m+1}\right) \geq v \Pr\left(\sum_{n=2}^{N} b(v_n) \geq c\right).$$

or

$$v \Pr\left(\sum_{n=2}^{N} b(v_n) = \frac{mc}{m+1}\right) \geq \frac{c}{m+1} \Pr\left(\sum_{n=2}^{N} b(v_n) \geq \frac{mc}{m+1}\right).$$

Since

$$\Pr\left(\sum_{n=2}^{N} b(v_n) = \frac{mc}{m+1}\right) = C_{N-1}^{m} (1 - F(a))^m F(a)^{N-1-m}$$

and

$$\Pr\left(\sum_{n=2}^{N} b(v_n) \geq \frac{mc}{m+1}\right) = \sum_{h=m}^{N-1} C_{N-1}^{h} (1 - F(a))^h F(a)^{N-1-h}$$

we conclude that $\phi\left(\frac{c}{m+1}\right) \geq \phi(0)$ if and only if $v \geq a$. ■

**Remark 2** *The above equilibrium is not ex-post efficient. In this equilibrium, if the good is provided then the sum of the player's valuations is at least* $(m+1)a > c$.

**Remark 3** *If every player is pivotal, i.e. if* $N = m+1$, *then the equilibrium (2) is given by:*

$$b(v) = \begin{cases} 0 & \text{if} \quad v < \frac{c}{m+1} \\ \frac{c}{m+1} & \text{if} \quad \frac{c}{m+1} \leq v \leq 1. \end{cases}$$

*Thus each player considers the cost equally divided among the players and contributes if and only if his value is at least his share. Efficiency requires that the good is provided whenever* $\sum v_i \geq c$ *whereas the above equilibrium requires each* $v_i$ *to be greater or equal to* $\frac{c}{m+1}$.

A natural question is whether the subscription game admits any equilibria that are efficient. We show below that every equilibrium of the subscription game is inefficient. We need two lemmas to prove the inefficiency result.

**Lemma 3** *Suppose player* $j, 2 \leq j \leq n$ *plays strategy* $b_j(\cdot)$ *and that* $b(\cdot)$ *is a best response for player* 1. *Then*

$$v > v' \Longrightarrow \Pr\left(\sum_{j=2}^{n} b_j(v_j) \geq c - b(v)\right) \geq \Pr\left(\sum_{j=2}^{n} b_j(v_j) \geq c - b(v')\right). \tag{4}$$

**Proof.** Define $H(v_{-1}) = \sum_{j=2}^{n} b_j(v_j)$, $b = b(v)$ and $b' = b(v')$. Then by the definition of $b(\cdot)$ we have

$$\begin{aligned} (v-b)\Pr(H(v_{-1}) \geq c - b(v)) &\geq (v-b')\Pr(H(v_{-1}) \geq c - b(v')) \text{ and} \\ (v'-b')\Pr(H(v_{-1}) \geq c - b(v')) &\geq (v'-b)\Pr(H(v_{-1}) \geq c - b(v)). \end{aligned}$$

Adding both inequalities and simplifying we have

$$(v-v')(\Pr(H(v_{-1}) \geq c - b(v)) - \Pr(H(v_{-1}) \geq c - b(v'))) \geq 0.$$

■

**Lemma 4** *Suppose* $\overline{v}$ *is such that* $\Pr\left(\sum_{j=2}^{n} b_j(v_j) \geq c - b(\overline{v})\right) > 0$. *Then for any* $v \geq v' \geq \overline{v}$ *we have* $b(v) \geq b(v')$. *Thus the optimal response* $b(\cdot)$ *is non-decreasing for any signal* $v$ *greater than some signal* $\overline{v}$ *that gives a positive probability of getting the public good.*

**Proof.** Suppose $v > v' > \overline{v}$. Then from (4) above we have

$$\begin{aligned} \Pr(H(v_{-1}) \geq c - b(v)) &\geq \Pr(H(v_{-1}) \geq c - b(v')) \\ &\geq \Pr(H(v_{-1}) \geq c - b(\overline{v})) > 0. \end{aligned}$$

Now note that this and the nonnegativity of the expected utility implies that $b = b(v) \leq v$ and $b' = b(v') \leq v'$. If $b > v'$ then $b > b'$. And if $b \leq v'$:

$$\begin{aligned} (v'-b')\Pr(H(v_{-1}) \geq c - b') &\geq (v'-b)\Pr(H(v_{-1}) \geq c - b) \\ &\geq (v'-b)\Pr(H(v_{-1}) \geq c - b'). \end{aligned}$$

In the last inequality above we used (4) again. Therefore $v' - b' \geq v' - b$ and $b \geq b'$. ■

**Remark 4** *The last two lemmas show that although an equilibrium strategy may not be non-decreasing it is non-decreasing only for any valuation greater than a valuation that yields a positive probability of obtaining the good. Also* $b(v) < v$ *in this case.*

Finally we can prove that all equilibria of the subscription game are inefficient.

**Theorem 5 (Subscription game inefficiency)** *Suppose the distribution* $F$ *has a density* $f(x) > 0,\ x \in [0,1]$. *Then the probability that the good is provided given that it is efficient to do so is uniformly less than one in the subscription game.*

**Proof.** Fix a $\zeta$, such that $\frac{c}{n} < \zeta < 1$. We want to show that there is a $R < 1$ such that for any equilibrium $(b^1(\cdot), \ldots, b^n(\cdot))$, the conditional probability

$$\Pr\left(\sum_{j=1}^{n} b^j(v_j) \geq c \,\middle|\, \sum_{j=1}^{n} v_j \geq c\right) \leq R. \tag{5}$$

We note that $b^i(v)$ is a solution of

$$\max_{x \geq 0} (v - x) \Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - x\right).$$

Suppose first that for some $i \leq n$, $\Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - b^i\left(\frac{c}{n}\right)\right) = 0$. Then it is clear from (4) that if $v_i \leq \frac{c}{n}$, $\Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - b^i(v_i)\right) = 0$ as well. Thus from Fubini's theorem it follows that

$$\Pr\left(\sum_{j=1}^{n} b^j(v_j) \geq c, \sum_{j=1}^{n} v_j \geq c\right) = \Pr\left(\sum_{j=1}^{n} b^j(v_j) \geq c, \sum_{j=1}^{n} v_j \geq c, v_i > \frac{c}{n}\right) \leq$$

$$\Pr\left(\sum_{j=1}^{n} v_j \geq c, v_i > \frac{c}{n}\right) \leq \max_{1 \leq i \leq n} \Pr\left(\sum_{j=1}^{n} v_j \geq c, v_i > \frac{c}{n}\right) < \Pr\left(\sum_{j=1}^{n} v_j \geq c\right).$$

Suppose now that for every $i \leq n$, $\Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - b^i\left(\frac{c}{n}\right)\right) > 0$. We now divide the proof in two cases.

**Case 1**: If $\sum_{j=1}^{n} b^j(\zeta) < c$.

Define

$$X = \left\{ v \in [0,\zeta]^n ; \sum_{j=1}^{n} v_j \geq c \right\}. \tag{6}$$

It is clear that $\Pr(X) > 0$ and since for every $v_j \leq \zeta$ it is true that $b^j(v_j) \leq \max\left\{ b^j\left(\frac{c}{n}\right), b^j(\zeta) \right\} = b^j(\zeta)$ then $\sum_{j=1}^{n} b^j(v_j) < c$ if $(v_1, \ldots, v_n) \in X$. Therefore

$$\begin{aligned}
\Pr\left( \sum_{j=1}^{n} b^j(v_j) \geq c, \sum_{j=1}^{n} v_j \geq c \right) &= \Pr\left( \sum_{j=1}^{n} b^j(v_j) \geq c, \sum_{j=1}^{n} v_j \geq c, v \notin X \right) \\
&\leq \Pr\left( \sum_{j=1}^{n} v_j \geq c, v \notin X \right) \\
&= \Pr\left( \sum_{j=1}^{n} v_j \geq c \right) - \Pr(X) \\
&< \Pr\left( \sum_{j=1}^{n} v_j \geq c \right).
\end{aligned}$$

**Case 2**: $\sum_{j=1}^{n} b^j(\zeta) \geq c$.

In this case if $v_j \geq \zeta$ for all $j \neq i$ then $b^j(v_j) \geq b^j(\zeta)$ for all $j \neq i$. Thus

$$\sum_{j \neq i} b^j(v_j) \geq \sum_{j \neq i} b^j(\zeta) = \sum_{j} b^j(\zeta) - b^i(\zeta) \geq c - \zeta.$$

Therefore

$$\Pr\left( \sum_{j \neq i} b^j(v_j) \geq c - \zeta \right) \geq \Pr(v_j \geq \zeta, j \neq i) = (1 - F(\zeta))^{n-1} =: \gamma.$$

Hence

$$\begin{aligned}
v - b(v) &\geq (v - b(v)) \Pr\left( \sum_{j \neq i} b^j(v_j) \geq c - b(v) \right) \\
&\geq (v - \zeta) \Pr\left( \sum_{j \neq i} b^j(v_j) \geq c - \zeta \right) \geq (v - \zeta)\gamma.
\end{aligned}$$

Thus

$$v(1-\gamma)+\gamma\zeta \geq b(v).$$

Define $Y = \{v \in [0,1]^n ; b^j(v_j) \leq v_j, \text{for all } j\}$. An application of Fubini's theorem shows that

$$\Pr\left(\sum_{j=1}^{n} b^j(v_j) \geq c, Y^c\right) = 0.$$

If $v \in Y$ and $\sum_{j=1}^{n} b^j(v_j) \geq c$ then $\sum_{j=1}^{n} v_j \geq c$. Thus

$$\Pr\left(\sum_{j=1}^{n} b^j(v_j) \geq c, \sum_{j=1}^{n} v_j \geq c\right) = \Pr\left(Y \cap \sum_{j=1}^{n} b^j(v_j) \geq c,\right)$$
$$\leq \Pr\left(\sum_{j=1}^{n} \min\{v_j, v_j(1-\gamma)+\gamma\zeta\} \geq c\right) < \Pr\left(\sum_{j=1}^{n} v_j \geq c\right).$$

Thus if we take

$$R = \max\left\{\begin{array}{l} \Pr\left(\sum_{j=1}^{n} \min\{v_j, v_j(1-\gamma)+\gamma\zeta\} \geq c\right), \\ \Pr\left(\sum_{j=1}^{n} v_j \geq c, v \notin X\right), \\ \max_{1\leq i\leq n} \Pr\left(\sum_{j=1}^{n} v_j \geq c, v_i > \frac{c}{n}\right), \end{array}\right\} \tag{7}$$

we finish the proof.

■

**Remark 5** *The hypothesis of positive density or even the existence of a density for the distribution $F$ can be avoided if $R$ defined in (7) is less than* 1.

# 6 The Contribution Game

In this section we analyze the contribution game i.e. where players make contributions that are not refunded if the threshold is not met. Players' values are determined by independent draws from a continuous distribution $F : [0,1] \to \mathbb{R}$. The cost of the public good is $c$.

If players $i = 2, 3, \ldots, N$ follow the function $b(\cdot)$ and player 1 with valuation $v$ contributes $x \geq 0$, his expected surplus is

$$\begin{aligned} \phi(x) &= v \Pr(x + b(v_2) + \ldots + b(v_N) \geq c) - x \\ &= v \Pr(b(v_2) + \ldots + b(v_N) \geq c - x) - x \end{aligned}$$

The following theorem shows that the coordination problem in the contribution game is so severe that if the cost of the public good is slightly above the aggregate mean of the valuations, then the unique equilibrium of the contribution game is for each player to contribute zero no matter what his value is.

Denote by $\mu$ the mean and by $\sigma^2$ the variance of the distribution $F$. Define $\gamma_N$ as the root $c$ of $N\sigma^2 - (N-1)\,c\left(\frac{c}{N} - \mu\right)^2 = 0$. Define also $R_N = \max\left\{\gamma_N, \mu(N-1) + 1 + \sigma\sqrt{N-1}\right\}$.

**Theorem 6 (Strong inefficiency)** *Suppose that players' valuations are distributed in the* $[0,1]$ *interval and the number of players is* $N$*. If the cost is* $c \geq R_N$*, then the unique equilibrium of the contribution game is* $b^j \equiv 0$.

**Proof.** Suppose $v < 1$ is Player's $i$ valuation. If $(b^1, \ldots, b^n)$ is an equilibrium, $i$'s expected utility from contributing $x$ is

$$\phi(x) = v \Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - x\right) - x.$$

Since $b^j(v_j) \leq v_j$ we have

$$\phi(x) \leq v \Pr\left(\sum_{j \neq i} v_j \geq c - x\right) - x \leq \Pr\left(\sum_{j \neq i} v_j \geq c - x\right) - x =: g(x)\,. \quad (8)$$

the inequality being strict if $\Pr\left(\sum_{j \neq i} v_j \geq c - x\right) > 0$. Thus it suffices to prove that if $1 \geq x \geq \frac{c}{N}$ then $g(x) \leq 0$. Since in this case the optimal bid $0 \leq b^i(v) < \frac{c}{N}$ and therefore $\sum_{j=1}^{N} b^j(v_j) < c$ for every $(v_1, \ldots, v_N)$. Hence $b^i(v_i) = 0$.

Suppose therefore that $x \geq \frac{c}{N}$. We then have

$$\Pr\left(\sum_{j \neq i} v_j \geq c - x\right) = \Pr\left(\sum_{j \neq i} (v_j - \mu) \geq c - x - (N-1)\mu\right)$$

$$\begin{aligned} &\leq \frac{E\left[\left(\sum_{j\neq i}(v_j-\mu)\right)^2\right]}{\left(c-x-(N-1)\,\mu\right)^2} \\ &= \frac{(N-1)\,\sigma^2}{\left(c-x-(N-1)\,\mu\right)^2}. \end{aligned}$$

It follows that

$$g(x) \leq \frac{(N-1)\,\sigma^2}{\left(c-x-(N-1)\,\mu\right)^2} - x =: h(x).$$

The function $h(\cdot)$ is strictly convex. Note that $h(1) = \frac{(N-1)\sigma^2}{(c-1-(N-1)\mu)^2} - 1 \leq 0$. Also

$$N(N-1)\left(\frac{c}{N}-\mu\right)^2 h\left(\frac{c}{N}\right) = N\sigma^2 - (N-1)\,c\left(\frac{c}{N}-\mu\right)^2 \leq 0.$$

Thus

$$\max_{\frac{c}{N}\leq x\leq 1} h(x) = \max\left\{h\left(\frac{c}{N}\right), h(1)\right\} \leq 0.$$

■

**Remark 6** *For example if $F$ is uniformly distributed on $[0,1]$, then $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$. Then if $N = 30$, $c \geq \max\{17.13, 17.05\} = 17.13$ the unique equilibrium of the contribution game is the zero equilibrium. If $N = 100$ the lower bound is $53.95$ if $N = 1000$ the lower bound is $512.75$. Moreover if $N \to \infty$, $\gamma_N/N \to \mu$.*

The above result implies that for a wide range of the cost of the public good, there is extreme underprovision when refunds are not allowed.

For small $N$ we can obtain sharper bounds on $R_N$. For example if $N = 2$, then $R_N = 1$. We turn to this in the next proposition. We show that the strategy "contributing zero" for both players is the only equilibrium for a general family of distributions.

**Proposition 7** *Suppose $1 < c < 2$. If the distribution function satisfies $F(z) \geq z$, $z \in (0,1)$ then the best response to $b_2(\cdot)$ such that $b_2(v_2) \leq v_2$ is $b_1(v_1) = 0$ .*

**Proof.** Define $\phi(x) = vP(b_2(v_2) \geq c-x)-x$. If $x > v$ then $\phi(x) \leq v-x < 0$. Thus to maximize $\phi$ we must have $x \leq v$. Suppose $x > 0$. If $x < c-1$ then $\phi(x) = v \cdot 0 - x = -x < 0$. Suppose now $c-1 \leq x \leq v$. We have $\phi(c-1) = -x$ and $\phi(v) \leq 0$. Now:

$$\begin{aligned} \phi(x) &\leq vP(v_2 \geq c-x) - x = v(1-F(c-x)) - x \\ &\leq v(1-(c-x)) - x = v(1-c) + (v-1)x < 0. \end{aligned}$$

■

Note that the above family of distributions includes well known distributions such as the uniform and the exponential distributions. More generally any concave distribution satisfies it.

**Remark 7** *We could prove also that there is no mixed strategy equilibrium. We omit the proof.*

The proposition (7) above is not true for every distribution. Intuitively if the distribution gives a high weight to high valuations then an equilibrium exists. This is the sense of the next proposition.

**Proposition 8** *Suppose that $c > 1$ and that there exists an $\alpha \in (0,1)$ such that $F\left(\frac{c}{2\alpha}\right) = 1-\alpha$. Then*

$$b(v) = \begin{cases} \frac{c}{2} & \text{if} \quad v \geq \frac{c}{2\alpha} \\ 0 & \text{if} \quad v < \frac{c}{2\alpha} \end{cases}$$

*is an equilibrium strategy of the contribution game with two players.*

We omit the proof but apply the result in an example.

**Example 1** *If $F(x) = x^\gamma$, then $\alpha$ is a solution of $\left(\frac{c}{2}\right)^\gamma = \alpha^\gamma(1-\alpha)$. Since*

$$\max\left\{\alpha^\gamma(1-\alpha); \alpha \in [0,1]\right\} = \left(\frac{\gamma}{1+\gamma}\right)^\gamma \left(\frac{1}{1+\gamma}\right),$$

*there is an equilibrium if $c \leq 2\left(\frac{\gamma}{1+\gamma}\right)^\gamma \frac{\gamma}{1+\gamma}$. Thus if $\gamma = 13$, $c = 1.5$ we have $\alpha = 0.88$ and $c/2\alpha \cong 0.85$.*

The previous theorems shows that for high enough cost (i.e. greater than $R_N$) the contribution game efficiency is zero. The following theorem shows that even for low costs, any equilibrium of the contribution game is inefficient.

Define $S_m(x) = \Pr\left(\sum_{j=1}^{m} v_j \leq x\right)$. The following theorem shows that the probability that the good is provided given that it is efficient to do so is bounded away from 1 in any equilibrium of the contribution game.

**Theorem 9 (Contribution game inefficiency)** *The efficiency probability of the contribution game is at most*

$$\gamma = \frac{\Pr\left(\sum_{i=1}^{n} v_i \left(1 - S_{n-1}(c - v_i)\right) \geq c\right)}{1 - S_n(c)} < 1.$$

**Proof.** : Suppose $(b^1(\cdot), \ldots, b^n(\cdot))$ is an equilibrium of the contribution game. It is clear that $b^i(v) \leq v$ for all $v$. Player $i$ with valuation $\bar{v}_i$ expected utility is

$$\bar{v}_i \Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - b^i(\bar{v}_i)\right) - b^i(\bar{v}_i) \geq \bar{v}_i \Pr(\sum_{j \neq i} b^j(v_j) \geq c) \geq 0.$$

Therefore

$$\bar{v}_i \Pr\left(\sum_{j \neq i} v_j \geq c - \bar{v}_i\right) \geq \bar{v}_i \Pr\left(\sum_{j \neq i} b^j(v_j) \geq c - b^i(\bar{v}_i)\right) \geq b^i(\bar{v}_i).$$

So $\bar{v}_i(1 - S_{n-1}(c - \bar{v}_i)) \geq b^i(\bar{v}_i)$ for every $i$ and every $\bar{v}_i$. Thus

$$\begin{aligned}
\Pr\left(\sum_{i=1}^{n} b^i(v_i) \geq c\right) &\leq \Pr\left(\sum_{i=1}^{n} v_i(1 - S_{n-1}(c - v_i)) \geq c\right) \\
&< \Pr(\sum_{i=1}^{n} v_i \geq c) \\
&= 1 - S_n(c).
\end{aligned}$$

■

# 7 The differential equation approach to find equilibrium strategies

The theory of single-object auctions with private independent values provides us with a method to find the (symmetric) equilibrium strategy that is very natural. Usually one supposes that bidders $i = 2, \ldots, N$ bid accordingly to the strictly increasing strategy $b(\cdot)$ and finds Bidder 1's best reply $x$ by means of the first order condition for expected utility maximization. Then in equilibrium $x = b(v)$ and we obtain a differential equation for $b(\cdot)$. Let us try to mimic this approach to find equilibria for the contribution and subscription games. Recall from the discussion following equation (1) that if $N > 2$, detailed information on $b(\cdot)$ is needed in order to derive the distribution of $\sum_{j \neq i} b(v_j)$. Let us therefore suppose $N = 2$. We suppose also that $f = F' > 0$ exists and is continuous everywhere. First we examine the subscription game.

## 7.1 The subscription game differential equation.

So suppose Player 2 bids accordingly to $b(\cdot)$ which is strictly increasing. Player 1's expected surplus given that he has a value $v$ and contributes $x \geq 0$ is given by

$$\begin{aligned}
\phi(x) &= (v - x) \Pr(b(v_2) \geq c - x) &\quad (9)\\
&= (v - x) \Pr\left(v_2 \geq b^{-1}(c - x)\right) &\quad (10)\\
&= (v - x)\left(1 - F\left(b^{-1}(c - x)\right)\right).
\end{aligned}$$

Note that we took the inverse of $b(\cdot)$ to pass from (9) to (10). Player 1's expected payoff is simply his surplus if the project is completed times the probability of completion. The first-order condition is

$$\phi'(x) = -\left(1 - F\left(b^{-1}(c - x)\right)\right) + (v - x) f\left(b^{-1}(c - x)\right)\left(b^{-1}\right)'(v - x) = 0.$$

In a symmetric equilibrium $x = b(v)$, so

$$(v - b(v)) f\left(b^{-1}(c - b(v))\right)\left(b^{-1}\right)'(c - b(v)) = 1 - F\left(b^{-1}(c - b(v))\right). \quad (11)$$

This is not an ordinary differential equation since the function $b(v)$ appears inside the argument. We will transform (11) in a system of two ordinary differential equations. Define $G(v) = b^{-1}(c - b(v))$. This implies that

$$\begin{aligned} b(G(v)) + b(v) &= c & (12) \\ (v - b(v)) f(G(v)) \left(b^{-1}\right)'(b(G(v))) &= 1 - F(G(v)). & (13) \end{aligned}$$

In (13) we used that $b(G(v)) = c - b(v)$. Since $b^{-1}(b(\omega)) = \omega$ we have that $(b^{-1})'(b(\omega))\, b'(\omega) = 1$. Thus choosing $\omega = G(v)$ we get

$$\left(b^{-1}\right)'(b(G(v))) = 1/b'(G(v)).$$

Substituting this in (13) it follows that

$$\frac{(v - b(v)) f(G(v))}{1 - F(G(v))} = b'(G(v)). \qquad (14)$$

Now from (12) applied to $v$ and to $G(v)$ we have that $b(G(v)) + b(v) = c = b(G^2(v)) + b(G(v))$. Hence $b(G^2(v)) = b(v)$ and therefore $G(G(v)) = v$. Therefore from (14) we conclude that

$$b'(v) = \frac{(G(v) - b(G(v))) f(v)}{1 - F(v)}. \qquad (15)$$

Differentiating (12) we obtain $b'(G(v))\, G'(v) + b'(v) = 0$. The following theorem describes the system of differential equations that an increasing equilibrium strategy to the subscription game must satisfy.

**Theorem 10** *The solution of the system of differential equations below characterizes a symmetric equilibrium involving strictly increasing strategies for the subscription game with two players with values determined by independent draws from a distribution $F : [0,1] \to \mathbb{R}$, where $F$ is continuously differentiable and $f = F' > 0$ everywhere.*

$$\begin{aligned} b'(v) &= \frac{(G(v) - b(G(v))) f(v)}{1 - F(v)}, & (16) \\ G'(v) &= -\frac{(1 - F(G(v)))(G(v) - b(G(v))) f(v)}{(1 - F(v))(v - b(v)) f(G(v))}. \end{aligned}$$

**Remark 8** *It is important to note that the system (16) gives only a candidate for an equilibrium strategy. An example in the appendix shows that some fine tuning may be necessary. This contrasts greatly with the solution we obtain in auction theory which is usually defined for every signal. Here it is possible to invert $b(\cdot)$ only locally. Moreover the existence of $G$ defined above is global in nature.*

## 7.2 The contribution game differential equation.

An analysis similar to the analysis made above gives the system of differential equations that a strictly increasing equilibrium of the contribution game must satisfy.

**Theorem 11** *The solution of the system of differential equations below characterizes a symmetric equilibrium involving strictly increasing strategies for the contribution game with two players with values determined by independent draws from a distribution* $F : [0,1] \to \mathbb{R}$*, where* $F$ *is continuously differentiable and* $f = F' > 0$ *everywhere.*

$$b'(v) = G(v)\, f(v), \tag{17}$$
$$G'(v) = -\frac{G(v)\, f(v)}{v f(G(v))}.$$

For example if $F$ is the uniform distribution,

$$\begin{aligned} b'(v) &= G(v), \\ G'(v) &= -\frac{G(v)}{v}. \end{aligned}$$

This gives $G(v) = \frac{k}{v}$ and $b(v) = a + k \log(v)$. Naturally $b(\cdot)$ cannot be an equilibrium for all $v$ since it is negative for small $v$. If $0 < c < 1/e$, $b(v) = \max\{0, c + k \log(v)\}$ where $k$ is such that $k^k = e^{-c}$, is an equilibrium.

# 8 Conclusion

We analyzed two mechanisms for private provision of discrete public goods with incomplete information and continuous contributions. The subscription mechanism refunds the money to contributors if the public good is not provided whereas the contribution mechanism does not allow for refunds.

Our analysis showed that, unlike the model with complete information, the coordination problem becomes more complex and efficient provision is no longer possible. Moreover, the two mechanisms lead to completely different outcomes. We showed that for the contribution game, for a wide range of costs of the public good, the good will never be provided. This confirms the evidence from the experimental literature of the superiority of subscription games over contribution games.

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# Appendix

## The subscription game with two players: The uniform distribution case

In what follows, we provide some intuition how to solve this problem when we take into account the boundary conditions. For the uniform distribution on the interval $[0,1]$, the unconstrained solution to the system of differential equations stated in Section 7.1 is given by the following pledging function:

$$b(v) = \frac{2c-1}{6} + \frac{v}{2}$$

Note, however, that in equilibrium a player may not follow $b(v)$ for any $v$ in $[0,1]$ as this may lead to pledging more than his value or more than the cost of the public good. Hence we need to impose the following boundary conditions

$$b(v) \leq c, \text{ (b1)} \qquad (18)$$
$$b(v) \leq v, \text{ (b2)} \qquad (19)$$
$$b(v) \geq 0, \text{ (b3)} \qquad (20)$$

It turns out that (b1) and (b3) are not binding since $c > 1$. Condition (b2) is binding as $b(v) > v$ for $v < \frac{2c-1}{3}$. We now show formally that the following is a symmetric equilibrium pledging strategy for the subscription game with two players whose values are uniformly distributed on $[0,1]$ and $1 < c < 2$:

$$b^*(v) = \begin{cases} \frac{2c-1}{6} + \frac{v}{2}, \text{ if } \frac{2c-1}{3} \leq v \leq 1, \\ 0, \text{otherwise.} \end{cases}$$

To verify that this is an equilibrium, we assume that Player 2 is following the proposed equilibrium pledging strategy, we have to find the best response of player 1. We first show that for $\frac{2c-1}{3} \leq v \leq 1$, $b_1(v) = \frac{2c-1}{6} + \frac{v}{2}$ is a best response to $b^*(v_2)$. Player 1's expected surplus, if he pledges $b$, is given by

$$\phi(b) = (v-b)\Pr(b + b^*(v_2) \geq c).$$

To find the maximum of $\phi$ first note that if $0 < c - b < b^*\left(\frac{2c-1}{3}\right) = \frac{2c-1}{3}$ then $\phi(b) = (v-b)\left(1 - \frac{2c-1}{3}\right) \leq \left(v - \frac{c+1}{3}\right)\left(1 - \frac{2c-1}{3}\right) = \phi\left(\frac{c+1}{3}\right)$. Note that if

$c-b \geq b^*(1)$ then $\phi(b) = 0$. Let us consider now $c - b^*(1) < b < c - b^*\left(\frac{2c-1}{3}\right)$. Then we have

$$\begin{aligned}\phi(b) &= (v-b)\Pr\left(\left\{v_2 \geq \frac{2c-1}{3}; v_2 \geq \frac{4c+1}{3} - 2b\right\}\right) = \\ & (v-b)\left(1 - \max\left\{\frac{2c-1}{3}, \frac{4c+1}{3} - 2b\right\}\right).\end{aligned}$$

There are two cases to consider:

**1)** $\frac{2c-1}{3} > \frac{4c+1}{3} - 2b$

In this case $\phi(b) = (v-b)\left(1 - \frac{2c-1}{3}\right) < \phi\left(\frac{c+1}{3}\right)$.

**2)** $\frac{2c-1}{3} \leq \frac{4c+1}{3} - 2b$

In this case $\phi(b) = (v-b)\left(1 - \frac{4c+1}{3} + 2b\right)$. This quadratic function has a unique maximum at $b^* = \frac{2c-1}{6} + \frac{v}{2}$. Thus $b^*$ is the optimal pledge if $b^* \in \left[c - b(1), c - b\left(\frac{2c-1}{3}\right)\right]$ and $\frac{2c-1}{3} \leq \frac{4c+1}{3} - 2b^*$. The last inequality is valid for all $v \in [0,1]$. The first inequality is valid if $v \in [\frac{2c-1}{3}, 1]$. Thus $b(v) = \frac{2c-1}{6} + \frac{v}{2}$ is the best response to $b^*(v_2)$ if $v \in [\frac{2c-1}{3}, 1]$. To finish let us find the best response for $v \in [0, \frac{2c-1}{3})$. It is clear from the reasoning in (2) above that the maximum of $\phi$ is not interior. Thus we need only to compare $\phi(c - b^*(1)) = \phi\left(\frac{2c-1}{3}\right) = \left(v - \frac{2c-1}{3}\right)\left(1 - \frac{4c+1}{3} + 2\frac{2c-1}{3}\right) = 0$ and $\phi\left(c - b^*\left(\frac{2c-1}{3}\right)\right) = \phi\left(\frac{c+1}{3}\right) = \left(v - \frac{c+1}{3}\right)\left(1 - \frac{2c-1}{3}\right) < 0$. Thus if $v \in [0, \frac{2c-1}{3})$ the maximum expected surplus is zero. Hence since pledging zero and pledging $\frac{2c-1}{3}$ give the same expected surplus, we finished the proof that $b(\cdot)$ is an equilibrium.

As the equilibrium pledging strategy is strictly increasing and differentiable in the relevant range our previous analysis is justified. We now provide the intuition for the solution of the subscription game and compare it to the equilibrium in a first price sealed-bid auction. Recall that in a symmetric equilibrium of a first-price sealed-bid auction — where the object is awarded to the individual with the highest bid — an individual bids in such a way to outbid the opponent with the highest value. That is, conditional on his value being the highest, his bid is equal to the expected value of the first-order statistics of his opponents.

In the subscription game, however, the good is provided to both players if their contributions add up to the cost of provision $c$. Thus, the problem becomes one of forecasting the lowest pledge one can make, given that it is below one's value, and still have the good being provided. Thus, the equilibrium pledging strategy implies that a player pledges the equivalent to the expected value of the other player being lower than his own, conditional on interval where pledges are less than or equal to the valuations for the public good.

Notice the distinction between the solution of the subscription game and the solution of the first-price auction. In the latter, if a bidder's value is not the highest, then in any symmetric equilibrium with increasing bids he will lose the object and therefore, in equilibrium, he does not have to consider what he would do if his value is not the highest one. In the subscription game this is not the case. If his value is the lowest of the two, he may still obtain the object and, thus, following (2) guarantees that this is the minimum pledge so that the object is provided and the players are sharing the cost in such way as to equalize their marginal contributions. This property of the equilibrium pledging strategies is very distinct from the result for first-price auctions and it captures the nature of the trade-off between the probability of the public good being provided and the free-riding behavior.

Next we compute the probability that the good will be provided whenever is efficient to do so in the symmetric equilibrium.

Since $b^*(v_1) + b^*(v_2) \geq c$ implies that $v_1 + v_2 \geq c$ we need to compute

$$\frac{\Pr\left(\{(v_1, v_2)\,; b^*(v_1) + b^*(v_2) \geq c\}\right)}{\Pr\left(\{(v_1, v_2)\,; v_1 + v_2 \geq c\}\right)}$$

Note that

$$\begin{aligned} & \Pr\left(\{(v_1, v_2)\,; b^*(v_1) + b^*(v_2) \geq c\ \}\right) \\ = \ & \frac{1}{2}\left(\frac{4-2c}{3}\right)^2 + \left(\frac{2c-1}{3} - (c-1)\right)\left(1 - \frac{c+1}{3}\right) \\ = \ & \frac{4}{3} - \frac{4}{3}c + \frac{1}{3}c^2 = \frac{1}{3}(c-2)^2 \end{aligned}$$

Moreover $\Pr\left(\{(v_1, v_2)\,; v_1 + v_2 \geq c\}\right) = \frac{1}{2}(2-c)^2$.

Therefore,

$$Pr\left(b^*(v_1) + b^*(v_2) \geq c | v_1 + v_2 \geq c\right) = \frac{2}{3}.$$