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MARCEL BERTINI RIBEIRO

ESSAYS ON IMPERFECT COMMON KNOWLEDGE IN MACROECONOMICS

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ABSTRACT

This dissertation study the implications of strategic uncertainty induced by imperfect common knowledge for macroeconomic models and economic policy.

In the first chapter, I evaluate whether central bank's transparency enhances the effectiveness of monetary policy. I study this question using a New Keynesian model in which firms do not observe the time-varying inflation target or monetary policy shocks. Two informational assumptions are considered: (i) firms observe the interest rate decisions only (standard assumption) and (ii) firms observe the interest rate and an idiosyncratic signal about the inflation target. Under the standard assumption, agents infer output and inflation fluctuations by realizing that other agents are acting exactly like them. That ceases to be true when agents face strategic uncertainty induced by the idiosyncratic signal. One key implication is that, in the case of a monetary contraction, greater transparency improves the inflation-output trade-off only under the second assumption.

In the second chapter, for a general class of DSGE models, I show that whenever agents extract information from endogenous variables that depends directly on the underlying unobserved shock, there is a qualitative difference in the signal extraction from those variables under imperfect information and imperfect common knowledge. This difference in learning about unobserved shocks does not vanish even in the limiting case when the variance of the private signal goes to infinity. Intuitively, strategic uncertainty prevents agents from knowing other agents' decision, despite that those actions are the same in equilibrium. This discontinuity challenges this benchmark assumption by showing the implicitly substantial knowledge about endogenous variables assumed available to agents under imperfect information.

The third chapter develops a novel solution method for a general class of DSGE models with imperfect common knowledge. The main contribution is that the method allows for the inclusion of endogenous state variables into the system of linear rational expectations equations under imperfect common knowledge. One key implication is that the endogenous persistence of state variables is the same under full information and imperfect common knowledge. A primer empirical evaluation of the informational frictions suggests that the model under imperfect common knowledge explains better the expectation data but is relatively worse at explaining the macroeconomic aggregates.

Keywords: Strategic uncertainty, higher-order expectations, imperfect common knowledge, transparency, inflation target, effectiveness of monetary policy, signal extraction; solution method.

RESUMO

Esta tese estuda as implicações da incerteza estratégica induzida pelo conhecimento comum imperfeito para modelos macroeconômicos e política econômica.

No primeiro capítulo, avalio se a transparência do banco central aumenta a eficácia da política monetária. Eu estudo essa questão usando um modelo Novo Keynesiano no qual as firmas não observam a meta de inflação variável no tempo e os choques de política monetária. Duas suposições sobre a informação dos agentes são consideradas: (i) as firmas observam apenas as decisões da taxa de juros e (ii) as firmas observam a taxa de juros e um sinal idiossincrático sobre a meta de inflação. Sob a suposição padrão, os agentes inferem flutuações de produto e inflação percebendo que outros agentes estão agindo exatamente como eles. Isso deixa de ser verdade quando os agentes enfrentam a incerteza estratégica induzida pelo sinal idiossincrático. Uma implicação principal é que, no caso de uma contração monetária, maior transparência melhora o trade-off inflação-produto apenas sob a segunda hipótese.

No segundo capítulo, para uma classe geral de modelos DSGE, mostro que sempre que os agentes extraem informações de variáveis endógenas que dependem do choque subjacente não observado, as extrações de sinal daquelas variáveis sob informação imperfeita e conhecimento comum imperfeito são diferentes. Essa diferença no aprendizado de choques não observados não desaparece nem no caso limite quando a variação do sinal privado vai para o infinito. Intuitivamente, a incerteza estratégica impede que os agentes conheçam a decisão de outros agentes, apesar de essas ações serem as mesmas em equilíbrio. Essa descontinuidade desafia a suposição padrão, mostrando o conhecimento substancial sobre variáveis endógenas implicitamente assumido disponível para agentes sob informação imperfeita.

O terceiro capítulo desenvolve um novo método de solução para uma classe geral de modelos DSGE com conhecimento comum imperfeito. A principal contribuição é que o método permite a inclusão de variáveis de estado endógeno no sistema de equações lineares de expectativas racionais sob conhecimento comum imperfeito. Uma implicação chave é que a persistência endógena de variáveis de estado é a mesma sob informação completa e conhecimento comum imperfeito. Uma avaliação empírica preliminar das fricções informacionais revela que o modelo sob informação imperfeita e dispersa explica melhor os dados de expectativas do que o modelo de informação completa. No entanto, isso ocorre ao custo de ser relativamente pior na explicação dos agregados macroeconômicos.

Palavras-chave: Incerteza estratégica, expectativas de ordens altas, conhecimento comum imperfeito, transparência, meta de inflação, efetividade da política monetária, extração de sinal, método de solução.

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Chapter 1

Inflation target expectations, transparency and monetary policy

Abstract

Does transparency increase the effectiveness of monetary policy? I study this question using a New Keynesian model in which firms do not observe the time-varying inflation target or monetary policy shocks. Two informational assumptions are considered: (i) firms observe the interest rate decisions only (standard assumption) and (ii) firms observe the interest rate and an idiosyncratic signal about the inflation target.

Under the standard assumption, agents infer output and inflation fluctuations by realizing that other agents are acting exactly like them. That ceases to be true when agents face strategic uncertainty induced by the idiosyncratic signal. Since inflation and output are primary determinants of the interest rate, this difference in information sets changes firms' ability to extract information from the interest rate. One key implication is that, in the case of monetary contraction, greater transparency improves the inflation-output trade-off only under the second assumption.

Keywords: Transparency, inflation target, higher-order expectations, effectiveness of monetary policy, signal extraction.
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1.1 Introduction

Over the last few decades, one of the most remarkable changes in monetary policy has been the increase in transparency and accountability. Central banks have broadly increased their communication about individual policy decisions by utilizing minutes and statements, in addition to more structured analysis. Another crucial aspect is the emphasis on the adoption of an explicit numerical target for inflation by several countries. These policy changes are based on the widespread belief among central banks that transparency enhances the effectiveness of monetary policy. For instance, in January 2012, the Federal Reserve decided to clarify that an inflation rate of two percent is consistent with its price stability goal in their dual mandate. The reasoning for this decision was made clear: "Such clarity facilitates well-informed decision-making by households and businesses, reduces economic and financial uncertainty, increases the effectiveness of monetary policy, and enhances transparency and accountability, which are essential in a democratic society"(FOMC; 2012).

This paper evaluates the claim that transparency increases the effectiveness of monetary policy by studying the effects of transparency about the goal of monetary policy on the trade-off between inflation and output. To address this question, I use the standard New Keynesian model in which central bank follows a Taylor rule with a time-varying, persistent, exogenous inflation target. This inflation target represents the goal of the central bank, which is not known by the private sector. I assume that firms do not observe changes in the target or monetary shocks and compare two informational assumptions: i) firms extract information about them from the interest rate, which I call imperfect common information (ICI), and ii) firms observe interest rate decisions and an idiosyncratic signal about the inflation target, which I call imperfect common knowledge (ICK), following the literature that has employed such an informational structure (see, for instance, Woodford; 2002; Adam; 2007; Nimark; 2008). Central bank communication is represented by noisy public signal about the inflation target such that its precision is interpreted as the degree of transparency (see Faust and Svensson; 2001, 2002).

This paper has two main contributions. First, I show that introducing a noisy idiosyncratic signal affects not only the weight given to both the interest rate and the private signal but also how agents extract information from interest rate decisions. Second, transparency has opposite effects on the effectiveness of the monetary policy depending on the information available to price setters (common or dispersed) and on which policy is evaluated (disinflation or usual monetary contraction). Specifically, for a disinflation policy (decrease in the inflation target), greater transparency implies a better trade-off between inflation and output under both informational assumptions. However, for monetary contraction (a positive monetary

shock), improved transparency entails a better trade-off only under the dispersed information model because, in the ICK model, firms have a qualitative different learning process about the inflation target.

In the model with imperfect but common information, it is common knowledge that all firms act in exactly the same manner (share the same information and solve the same problem). This allows them to observe the price level and aggregate output. Then, since they know the monetary rule, information is extracted from the interest rate adjusted by the endogenous response to those variables. This learning process is a dynamic version of the standard signal extraction problem in which firms are unable to distinguish changes in the inflation target from monetary shocks. Thus, when firms observe a higher (adjusted) interest rate, their expectation is that it is a result of either a positive monetary shock or a lower inflation target.

The signal extraction under the dispersed information model is more complex. The idiosyncratic signal induces strategic uncertainty, i.e., firms do not know other firms' decisions in equilibrium. This prevents them from knowing current inflation and output once they are incapable of inferring the aggregate decisions from their own. Thus, strategic uncertainty affects the ability to extract information from the interest rate. In other words, when forming expectations about shocks from the observed interest rate, which obeys a Taylor rule, firms also have to consider the endogenous effects of the inflation target and monetary shocks (and their higher-order expectations) on the interest rate through the responses of inflation and output (which are known in the imperfect information case). This additional uncertainty serves as a confounding factor in the firms' signal extraction based on the Taylor rule.

The ICK model implies in a more realistic learning process. First, since firms have different information sets, they disagree in their inflation expectations both regarding short and long-term expectations, which is a common feature in inflation expectations data. Second, the learning process is intrinsically related to the dynamics of the model. The uncertainty regarding inflation and output translates into an additional factor that agents have to take into account when trying to understand the reasons why the interest rate have changed. Under imperfect information, the signal extraction depends only on the characteristics of the shocks and not on how they affect the economy.

As an example, suppose that firms observe an increase in the interest rate owing to a monetary shock. They have to consider not only the direct effect of shocks (as in the ICI model) but also whether this increase is a response to higher inflation and/or output, which, in turn, could be a response to a negative monetary shock or a higher target. Therefore, the signal extraction problem must balance the endogenous and direct effects of the shocks on

interest rate, which are affected by shocks in opposite directions. Since the inflation target process is very persistent, a change in the target has strong effects on endogenous variables. The opposite is true for monetary shocks since they are temporary. Hence, the endogenous effect is dominant for the former and dominated for the latter. This implies that if firms observe a higher interest rate after a positive monetary shock, they expect a higher inflation target – the opposite situation as in the ICI model.

Learning differences about the inflation target play an essential role in the inflation-output trade-off. Intuitively, since target shocks are very persistent, they have strong effects on inflation expectations, implying sizable shifts in the Phillips curve. Differences in learning about monetary shocks have a small role since they entail smaller shifts owing to their temporary nature. Interestingly, in the case of a monetary shock in the ICK model, since firms expect a increase in the target, there is a worse trade-off between inflation and output than in the full information model. The opposite is true in the ICI model: firms expect a lower target which implies a better trade-off. For the case of the target shock, the differences among the ICK and ICI models, regarding expectations about the monetary shock, are not sufficient to change the trade-off. Specifically, both models have a worse inflation-output trade-off than the full information model.

Building on the contributions of Nimark (2008) and Melosi (2017), I propose a solution method for the imperfect common knowledge model that includes exogenous public signals into agents' information sets. Within this framework, a lower standard deviation of the public signal about the target represents a higher degree of transparency of the central bank. Therefore, by providing a more accurate public signal about the target, greater transparency affects the economy through agents' expectations. Firms can better assess whether the change in the interest rate was a response to monetary shocks or a change in the inflation target.

Greater transparency implies that both models are closer to the full information model, but it does not change the qualitative differences in expectations formation. Therefore, transparency about the target implies lower disinflation costs, independent of the assumptions considered because both models have a worse inflation-output trade-off than the full information counterpart. For monetary shocks, however, since the trade-off in the ICK model is worse than in the full information model, higher transparency implies a better trade-off. Precisely the opposite happens in the ICI model since it entails a better trade-off between inflation and output for this particular shock.

This paper is organized as follows. Section 3.1 discuss the literature regarding changes in monetary policy rules, central bank transparency and imperfect common knowledge. Section 3.3.1 presents the models and discusses their differences in the signal extraction problem due

to the informational assumptions. Then, sections 1.4 and 1.5 relate the signal extraction differences to the trade-off between inflation and output and the effects of transparency, respectively. Section 1.6 concludes the paper.

1.2 Related literature

This section presents a brief review of the related literature. In subsection 1.2.1, I relate the model to the literature that studies shifts in monetary policy rules. Then, subsection 1.2.2 relates the model to the central bank transparency literature. Finally, in subsection 1.2.3, I discuss the literature regarding the imperfect common knowledge model.

1.2.1 Changes in monetary policy rules

Several papers study changes in monetary policy rules and their impact on inflation dynamics and the overall economy. Clarida et al. (2000) and Lubik and Schorfheide (2004) present evidence that US monetary policy authority has increased the interest rate response to inflation since the beginning of the 1980s. Both argue that before this change, monetary policy was not sufficiently reactive to inflation fluctuations and thus the central bank was unable to stabilize inflation. Therefore, the change in monetary policy is the main reason for the inflation stabilization in the 1980s.

Two main aspects of this interpretation have been questioned. First, the influential paper from Sims and Zha (2006) shows that changes in monetary policy are mainly stochastic (and not a deterministic one-time change) and that regime changes in the volatility of shocks are more important for explaining the stabilization of inflation. Second, Cogley et al. (2010) do not see the parameters of the Taylor rule as the determinant of the stabilization of inflation but instead consider changes in the inflation target to be the key explanation. Moreover, Schorfheide (2005) assumes that changes in monetary policy can be described as a Markov-switching regime in the inflation target and considers the cases in which the current regime is known and that in which it is not; in the latter case, agents use Bayesian updating to infer the monetary policy regime.

Ireland (2007) discusses the sources of such changes by estimating a DSGE model that allows the inflation target to be a response to structural temporary supply shocks in addition to exogenous changes. He finds evidence that during the high-inflation periods of the 1970s, transitory supply shocks had persistent effects on inflation owing to unwillingness of the Federal Reserve to incur output costs.

More related to this study, Erceg and Levin (2003) and Andolfatto et al. (2008) work with a model in which agents cannot disentangle persistent shifts in the inflation target from transitory shocks to the monetary policy rule. Thus, agents form rational expectations given their observations of interest rate decisions. In this case, inflation has an inertial behavior as a result of the persistence of agents' optimal forecast of unobserved quantities. This has important implications for the costs of disinflation and the existence of persistence in the inflation forecast errors. Therefore, inflation persistence is not an intrinsic feature of the economy but rather depends on the perception about the current monetary stance, i.e., the expectations about the inflation target and monetary shocks.[1] I show that central bank communication about the target can diminish the uncertainty about the target, implying a more accurate and faster learning process about the target and thus leading to lower output costs since the inflation expectations change accordingly.

1.2.2 Central bank transparency

Since the increases in transparency by central banks, the topic has drawn a lot of attention, particularly in the empirical literature.[2] Here, I will briefly discuss the theoretical literature. Since most economic decisions are forward-looking, communication and transparency are essential tools for central banks (see, for instance, Woodford; 2001, 2005). More information for agents can diminish uncertainty, thereby leading to more accurate and better decisions.

Most of the theoretical literature regarding central bank transparency builds on the Barro-Gordon framework, which focuses on the inflation bias and reputation issues. Geraats (2002) surveys the main results of this literature in which transparency (opacity) is usually defined as symmetric (asymmetric) information about the issue in question, e.g., the inflation target. In this literature, typically, the central bank is either fully transparent – e.g., it announces the inflation target – or fully opaque. Faust and Svensson (2001, 2002) are exceptions that allow a degree of transparency depending on the precision of the noise in the public signal announced by the central bank. This paper follows this approach by introducing a public signal about the inflation target.[3]

Geraats (2002) distinguishes two effects of transparency. First, asymmetric information implies higher uncertainty for agents that have an informational disadvantage, called "uncertainty effects". Second, there exists an "incentive effect", which is the central bank's willingness to manipulate the beliefs of others through signaling. Transparency will influence

[1] Inflation persistence can also depend on expectations regarding other shocks that affect inflation but are not related to monetary policy.
[2] Blinder et al. (2008) presents a comprehensive survey of the empirical findings.
[3] In Faust and Svensson (2001, 2002), the public signal is about the control error of inflation.

both effects, diminishing (vanishing) the uncertainty effect, which is welfare-improving, and mitigating the incentive effect, which may or may not be welfare-improving.[4]

More related to the modeling approach used in this paper, Morris and Shin (2002) show that agents whose actions are strategic complements and decided based on both public and private information put a disproportionately high weight on public information. Thus, sufficiently imprecise public information could generate undesired volatility in agents' decisions, which might be welfare-reducing. Svensson (2006) argues that in practice, this result is pro-transparency since public information is likely to be more accurate than private information. As in Morris and Shin (2002), the solution method proposed in this paper allows agents to observe both private and public signals, but here agents learn about dynamic fundamentals.

Another argument against central bank transparency from Morris and Shin (2005) is that higher transparency implies less informationally efficient markets. The idea is that when more public information about the economy is available, market prices and interest rates – which depend on expectations about the state of the economy – will respond less to the actual state of the economy but rather more strongly reflect information about the public information itself. Therefore, equilibrium market outcomes provide less information about economic fundamentals.

This paper focuses on the transparency about the inflation target within the New Keynesian framework, which now is the benchmark model for monetary policy evaluations. Eusepi and Preston (2010) and Ascari et al. (2017) also use the New Keynesian model to study the implications of transparency, but they emphasize the effects on the expectations stability properties of the model.[5] Moreover, while the former considers transparency both about the policy rule and the inflation target, the latter considers only transparency about the rule. Here the policy rule is common knowledge, but agents do not observe its shocks.

[4] Geraats (2002) shows an example in which a central bank being transparent about the target eliminates uncertainty about inflation and output but also removes the reputation considerations that reduce the inflation bias. It turns out that the latter effect dominates and thus opacity is desirable in this example.

[5] Those papers use the expectations stability concept from Evans and Honkapohja (2001) that evaluates whether the learning process converges to the one implied under full information rational expectations. In that literature, agents do not know some parameters of the model. They behave as econometricians and compute expectations according to a reduced-form model. In the learning process applied in this paper, agents have rational but limited information. The learning process is about unknown stochastic processes instead of parameters.

1.2.3 Imperfect common knowledge

This literature introduces an idiosyncratic signal about an unobserved fundamental. This framework contains the usual perfect information model (common knowledge of information) as a particular case when the variance of the idiosyncratic signal goes to zero. Therefore, this imperfect and dispersed noisy information structure is usually referred to as imperfect common knowledge (ICK) (see, for instance Woodford; 2002; Adam; 2007; Nimark; 2008).

A growing body of literature uses this information setup to explain business cycles (see, e.g., Lorenzoni; 2009; Angeletos and La'O; 2010, 2013), to understand the real effects of monetary policy (Woodford; 2002) and to study optimal monetary policy (see Lorenzoni; 2010; Adam; 2007 and Paciello and Wiederholt; 2013 for models that include rational inattention). Angeletos and La'O (2011) also study optimal monetary policy under nominal rigidities within a more general setup for informational frictions.

Closer to this paper, Nimark (2008) studies a model which the main feature is imperfect common knowledge about the aggregate real marginal cost with sticky prices. Specifically, firms do not observe other firms' marginal cost. Since there are strategic complementarities in pricing decisions, firms must forecast other firms' pricing decisions to make their own decisions.

I propose a model with ICK about the inflation target such that agents observe interest rate decisions and their signal to predict other firms' forecast. Melosi (2017) studies the signaling channel of monetary policy in the same manner, but in his model, agents learn about supply, demand and, monetary shocks. Instead, I consider that firms extract from the interest rate information about changes in the inflation target and monetary shocks.

This paper compares the signal extraction problem under the dispersed and noisy information model proposed with the one from the imperfect information model. The latter is a fully forward-looking version of the models of Andolfatto et al. (2008), Erceg and Levin (2003) and Del Negro and Eusepi (2011). I show that since agents are extracting information from the interest rate, the learning processes under those assumptions are qualitatively different. Those differences have important implications on the effects of transparency of the central bank on the inflation-output trade-off.

I also contribute to the literature by proposing a solution method that includes exogenous public signals into agents' information set. This solution method builds on the contributions of Nimark (2008) and Melosi (2017). The solution is a dynamic version of the static model with idiosyncratic and public signals from Morris and Shin (2002).

1.3 Model

Apart from the introduction of an imperfectly observed inflation target, the framework in the following is a standard New Keynesian model with sticky prices and monopolistic competition. The economy is populated by a representative household, a continuum of monopolistic competitive firms and a monetary authority. The central bank chooses the interest rate following a Taylor rule with a time-varying target that is not observed by the private sector. Firms choose their price taking into account the probability of not being able to readjust prices subject to their information. I consider two different informational assumptions: i) imperfect common information, where firms observe the interest rate chosen by the central bank and form expectations about the target and the monetary shock based only on this information, and ii) imperfect (common) knowledge, where agents observe both the interest rate and an idiosyncratic signal about the inflation target. For simplicity, perfect information for households is assumed.[6]

The model has the following timing protocol. Every period t is divided into two stages, in which actions are taken simultaneously. In stage 1, the inflation target and monetary shocks realize, the central bank sets its interest rate and, in the ICK model, firms observe their idiosyncratic signal s_{it}. In stage 2, given the observables, firms set their price $P_{i,t}$ and hire labor $L_{i,t}$ to produce and deliver the demanded quantity, $C_{i,t}$. Households also decide their consumption composite C_t, labor supply L_t and demand for one-period nominal bonds, B_t, and markets clear.

1.3.1 Households

There is a representative household whose utility function is given by

$$U = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(\frac{C_\tau^{1-\gamma}}{1-\gamma} - \frac{L_\tau^{1+\varphi}}{1+\varphi} \right), \tag{1.1}$$

where $0 < \beta < 1$ is the discount factor, γ is the inverse of the intertemporal rate of substitution, and φ is the inverse of the Frisch labor supply elasticity. For simplicity, I assume that households have full information with expectation operator denoted by $\mathbb{E}_t$.

The composite consumption good is a Dixit-Stiglitz aggregator $C_t = \left(\int_0^1 (C_{i,t})^{(\varepsilon-1)/\varepsilon} di \right)^{\varepsilon/(\varepsilon-1)}$, where $C_{i,t}$ is the consumption of each variety i in period t and $\varepsilon > 1$ is the elasticity of substitution between varieties. The household budget constraint is given by

[6] This assumption do not affect the results qualitatively.

$$P_t C_t + B_t = W_t L_t + T_t + (1 + I_{t-1}) B_{t-1}. \tag{1.2}$$

where B_t is the quantity of one-period nominal bond that the household holds in period t that pays the interest rate I_t. L_t denotes the hours of labor of the representative household that receives the hourly nominal wage, W_t, and T_t is the nominal net transferences.

The household problem can be divided into two parts. First, she chooses her optimal allocation of consumption between varieties by maximizing the consumption bundle, C_t, subject to the consumption budget. Therefore, the demand for each good i is given by

$$C_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} C_t, \tag{1.3}$$

where the price level of the composite good is given by $P_t = \left(\int_0^1 (P_{i,t})^{(1-\varepsilon)} di\right)^{1/(1-\varepsilon)}$. Second, the household chooses her optimal allocation between bonds, labor, and consumption that maximizes the utility function subject to the household budget constraint.

1.3.2 Monetary authority

The monetary authority sets the interest rate according to a log-linearized Taylor-rule type of reaction function given by

$$i_t = \pi_t^* + \phi_\pi(\pi_t - \pi_t^*) + \phi_y y_t + \eta_t, \tag{1.4}$$

where π_t is the (gross) inflation, y_t denotes the output, and η_t is a monetary shock. Henceforth, all lowercase variables refer to the log-deviation of the respective uppercase variables from their respective steady-state values. Therefore, $\pi_t^* = log(\Pi_t^*) - log(\Pi^*)$ is the log-deviation of the inflation target from its steady-state value, Π^*, which is constant and common knowledge.

To avoid indeterminacy issues, the response to inflation is chosen such that $\phi_\pi > 1$. Furthermore, I assume that the log-linearized inflation target and monetary shock follow autoregressive processes:

$$\pi_t^* = \rho_\pi \pi_{t-1}^* + \varepsilon_t^\pi \tag{1.5}$$

$$\eta_t = \rho_\eta \eta_{t-1} + \varepsilon_t^\eta \tag{1.6}$$

where $\varepsilon_t^j \sim \mathcal{N}(0, \sigma_j^2)$, for $j = \{\eta, \pi\}$.

Note that the assumed reaction function and the inclusion of an inflation target do not necessarily mean that the model applies only to inflation targeting regimes. It describes

any central bank that seeks to offset the developments of inflation and output by increasing the interest rate based on an implicit target that can vary over time. As discussed in the literature review, there is no definitive method for modeling changes in the monetary policy rule. I follow the recent literature that models changes in monetary policy as a monetary rule with a time-varying exogenous inflation target (see Erceg and Levin; 2003; Ireland; 2007; Andolfatto et al.; 2008; Cogley et al.; 2010; Del Negro and Eusepi; 2011).

One alternative to the proposed approach is to let the Taylor rule coefficients (ϕ_π and/or ϕ_y) vary over time and adopt a fixed inflation target. However, as pointed out by Schorfheide (2005), changes in the inflation target are important up to a first-order approximation, whereas changes in the response of the interest rate to deviations of inflation from the inflation target (and/or output) are a second-order effect.[7] Moreover, there is evidence that inflation has a very smooth trend whose variation explains most of the unconditional variance of inflation (see the evidence for the US presented in Stock and Watson (2007) and Cogley et al. (2010); see Garnier et al. (2015) for 14 advanced economies). Assuming a very persistent process for the inflation target is a straightforward method to incorporate this behavior of the trend inflation.

Previous literature, e.g., Ball (1994a) and Mankiw and Reis (2002) define a disinflation episode as permanent changes in money supply growth. Since the monetary authority is setting the interest rate, disinflation is a permanent drop in the inflation target. I follow Erceg and Levin (2003) by modeling the target as a very persistent but still stationary process.

1.3.3 Pricing decisions and information

There is a continuum $i \in [0, 1]$ of monopolistic competition firms that produce a differentiated good Y_{it} using a linear production function with labor as its only input such that

$$Y_{it} = L_{it}^{\alpha}, \tag{1.7}$$

where $\alpha \in (0, 1]$. Introducing decreasing returns into the production function increases the so called "real rigidities" that help pricing decisions to be strategic complements. As in Calvo (1983), in each period, there is a constant probability $(1 - \theta)$ that each firm will re-optimize its price. Those firms that do not optimize their prices are assumed to adjust mechanically

[7] This is not inconsistent with the arguments of Clarida et al. (2000) and Lubik and Schorfheide (2004), since those authors argue for a one-time permanent change in the coefficients of the Taylor rule. Temporary changes in those coefficients are a second-order effect.

their prices by the steady-state inflation, Π^*.[8] This assumption implies that the evolution of the price index is given by

$$P_t = \left[\theta(\Pi^* P_{t-1})^{1-\varepsilon} + (1-\theta)\int_0^1 (P_{it}^*)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}, \tag{1.8}$$

where P_{it}^* denotes the optimal price that firm i is able to choose. Firms maximize the expected discounted value of their profit, which is given by

$$E_{i,t}\left[\sum_{s=0}^{\infty}(\beta\theta)^s \Lambda_{t,t+s}\left((\Pi^*)^s P_{it}^* - P_{t+s} MC_{i,t+s}\right) Y_{i,t+s}\right], \tag{1.9}$$

subject to the market clearing condition, $Y_{i,t} = C_{i,t}$, the firm individual firm demand (2.2) and the production function (2.A.5). $\Lambda_{t,t+s}$ is the stochastic discount factor, $MC_{i,t+s}$ is the real marginal cost, and $E_{it}[\cdot] \equiv E[\cdot|\mathcal{I}_t^i]$ is the expectation operator of firm i conditional on the current information set, $\mathcal{I}_t^i$.

The full information model is compared with two type of informational assumptions. In the ICI model, firms observe only the interest rate decision. Therefore, they will extract information from the Taylor rule (3.68) to form expectations about the inflation target and the monetary shock. Meanwhile, in the ICK model, firms observe, in addition to the interest rate, an idiosyncratic signal about the inflation target given by

$$s_{it} = \pi_t^* + v_{it}, \tag{1.10}$$

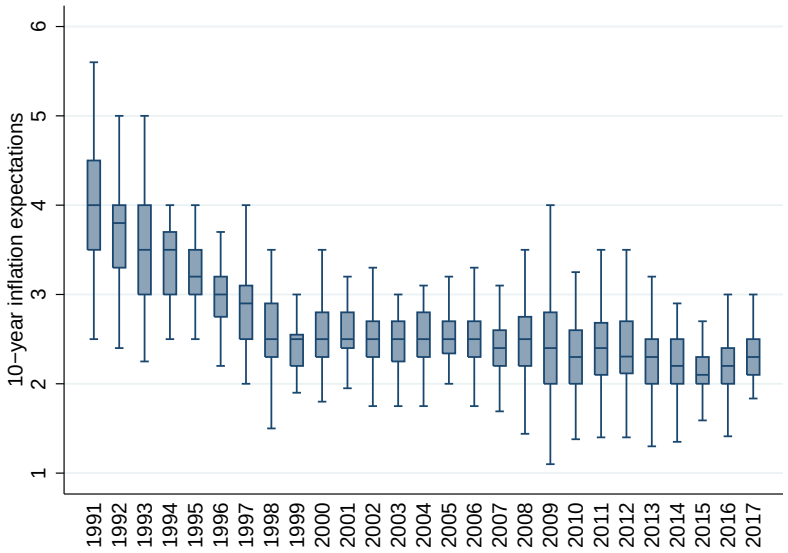
where $v_{it} \sim \mathcal{N}(0, \sigma_v^2)$. In this context, the interest rate works as an (endogenous) public signal since all firms share this observation.

Firms observe their idiosyncratic signal and do not communicate among themselves. This implies that there is disagreement in expectations even in the steady-state. Figure 1.1 shows the cross-sectional distribution of the 10-year ahead inflation expectations (CPI for the US) from the Survey of Professional Forecasters (SPF) over time during 1991 to 2017.[9] By observing the figure, two features of the data draw attention. First, the long-term expectations gradually decreased during the 1990s and became stable in 2000s and afterward. This feature can be consistent with persistent changes in the inflation target that I advocate here. Second, and more importantly, there is significant disagreement in the data. By definition, the ICI

[8] This type of indexing mechanism prevents the steady-state inflation from impacting the steady-state output level owing to a mechanical effect on the relative prices of re-optimizing firms vis-a-vis the firms not able to adjust. See Ascari (2004) for details.

[9] The Survey of Professional Forecaster data for 10-year ahead expectations data has limited availability in comparison with short-term expectations about the same price indexes. The longest data available is for the CPI inflation, which starts in Q4:1991.

Figure 1.1: Distribution of the 10-year ahead inflation expectations



Note: Individual data for the 10-year ahead CPI inflation from the Survey of Professional Forecasters. Data is quarterly, but the percentiles are calculated pooling observations from same year. The year of 1991 uses only data of the last quarter, and the year of 2017 uses only the first three quarters due to availability. The boxplot excludes outlier values.

model cannot account for this since all agents share the same information. The introduction of the idiosyncratic signal allows the ICK model to be consistent with disagreement in long-term expectations presented in the data.[10]

In the following, I present the differences in the solution and expectations formation under such assumptions about the information available to firms. Then, I discuss the implications for the trade-off between inflation and output and the effects of transparency.

1.3.4 Log-linearized model and solution

The household's optimization solution is the standard Euler equation. Its log-linearized version is given by

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\gamma}\left[i_t - \mathbb{E}_t \pi_{t+1}\right], \tag{1.11}$$

[10]By the interquartile range implied in the boxplot, one can see that the dispersion changes over time. The ICK model cannot account for that feature since the disagreement in the model is constant. Mankiw et al. (2004) document similar time-varying dispersion for short-term expectations data and show that the sticky information model can account for this feature.

where the aggregate constraint that $y_t = c_t$ was used. y_t denotes the output, i_t is the one-period (gross) interest rate, and π_t is the (gross) inflation rate. The log-linearized version of the price level equation (1.8) is

$$p_t = \theta p_{t-1} + (1-\theta)p_t^*, \tag{1.12}$$

where p_{it}^* is the optimal price of firm i and $p_t^* = \int_0^1 p_{it}^* di$ is its average. The log-linearized solution of the firms' problem is

$$p_{i,t}^* = (1-\beta\theta)E_{i,t}\left[\sum_{s=0}^{\infty}(\beta\theta)^s\left(p_{t+s} + \xi y_{t+s}\right)\right], \tag{1.13}$$

where $\xi = \frac{\alpha(\gamma-1)+1+\varphi}{\alpha+(1-\alpha)\varepsilon}$ is a parameter that controls the strategic complementarity (or substitutability) of firms' price decisions.[11] Therefore, in order to choose their optimal prices, firms use their individual information to forecast the paths of y_t and p_t

Using equations (2.1) and (1.12) and after some algebra, one can obtain the imperfect common knowledge New Keynesian Phillips Curve (ICK-NKPC):

$$\pi_t = (1-\theta)E_t^{(1)}\left[\pi_t\right] + \beta\theta E_t^{(1)}\left[\pi_{t+1}\right] + \kappa\theta E_t^{(1)}\left[y_t\right], \tag{1.14}$$

where $\kappa = \xi\frac{(1-\beta\theta)(1-\theta)}{\theta}$ and $E_t^{(1)}[\cdot] = \int_0^1 E_{it}[\cdot]\,di$ is the average expectation operator. Therefore, the equilibrium responses of the endogenous variables satisfy the Euler equation (1.11), the Phillips curve (1.14) and the Taylor rule (3.68).

In the case in which firms have full information, i.e., $E_{i,t}[\cdot] \equiv \mathbb{E}_t[\cdot]$, equation (1.14) becomes the standard New Keynesian Phillips curve, $\pi_t = \beta\mathbb{E}_t\left[\pi_{t+1}\right] + \kappa y_t$. In the ICI model, the current inflation and output are known – a feature that I emphasize latter – and thus, equation (1.14) becomes the imperfect information Phillips curve, which is given by

$$\pi_t = \beta E_t\left[\pi_{t+1}\right] + \kappa y_t, \tag{1.15}$$

where firms have a common expectation $E_t[\cdot] \equiv E[\cdot|\mathcal{I}_t]$ since they have the same information set, $\mathcal{I}_t$.

The solution of the imperfect information model is given by the matrices Q_1 and Q_2 such that

$$Y_t = Q_1 x_t + Q_2 E_t\left[x_t\right], \tag{1.16}$$

where $Y_t = [y_t \quad \pi_t \quad i_t]'$ is the vector of endogenous variables and $x_t = [\eta_t \quad \pi_t^*]'$ is the vector

[11] See Woodford (2003) for a detailed discussion.

of unobserved shocks.[12]

Finally, in the ICK model, each firm uses the information from the interest rate and its own signal to choose prices. The idiosyncratic information induces strategic uncertainty, i.e., firms do not know other firms' pricing decisions since they do not share the same information. Nimark (2008) shows that under this environment, by taking higher-order average expectations and substituting them iteratively, equation (1.14) can be rewritten as

$$\pi_t = \kappa\theta \sum_{k=1}^{\infty}(1-\theta)^{k-1}E_t^{(k)}[y_t] + \beta\theta \sum_{k=1}^{\infty}(1-\theta)^{k-1}E_t^{(k)}[\pi_{t+1}], \tag{1.17}$$

where $E_t^{(k)}[.]$ denotes the average k-th order expectation defined by $E_s^{(k)}[z_t] = \int_0^1 E_{is}\left[E_s^{(k-1)}[z_t]\right]di$ for $k \geq 1$, any periods $s \leq t$ and any variable z_t. For convenience, I define $E_s^{(0)}[z_t] = z_t$ such that the first-order average expectation is as defined above. Equation (1.17) emphasizes the well-known result that higher-order expectations matter when there are strategic complementarities and strategic uncertainty.

Nimark (2008) proposes a method that provides an approximate solution with a finite number of orders of expectations. The solution can be arbitrarily accurate since the effect of higher-order expectations on inflation decrease as the order increases.[13] Therefore, there is a matrix Q such that the model solution is given by

$$Y_t = Qx_t^{(0:\bar{k})}, \tag{1.18}$$

where $x_t^{(0:\bar{k})}$ denote the expectations hierarchy, $x_t^{(0:\bar{k})} \equiv \left[\begin{array}{cccc} x_t' & E_t^{(1)}[x_t]' & \cdots & E_t^{(\bar{k})}[x_t]' \end{array}\right]'$, the vector that stacks the average higher-order expectations from order 0 to $\bar{k}$.

Then, one can guess (and verify) that the dynamics of the expectations hierarchy can be defined as

$$x_t^{(0:\bar{k})} = Ax_{t-1}^{(0:\bar{k})} + B\varepsilon_t, \tag{1.19}$$

where $\varepsilon_t = [\varepsilon_t^{\eta} \quad \varepsilon_t^{\pi}]'$. Following Nimark (2008), in the Appendix 1.A.3, it is shown how the guessed matrices A, B and Q are pinned-down in equilibrium, and an algorithm to determine the fixed point implicit in the interaction of the expectations hierarchy (solution for A and B) and the response of endogenous variables to the hierarchy (solution for Q) is presented.

In the following, I show how expectations are formed in both models and highlight the

[12] An straightforward method to implement the solution of the ICI model is to write it as in the standard linear system of (full information) rational expectation equations considering the expectation $E_t[x_t]$ and x_t as correlated shocks. Then, the dynamics of x_t and $E_t[x_t]$ defined in the following by equations (1.22) and (1.23), respectively. See Blanchard et al. (2013) for similar approach.

[13] In equation (1.17), the coefficients of the k-th order expectation for the output and next-period inflation depend on $(1-\theta)^{k-1}$, which is decreasing in k.

differences in learning and their implications.

1.3.5 Signal extraction and expectations formation

A vast body of literature has studied the implications of dispersed information and higher-order expectations for business cycles and monetary policy. Less attention has been paid to the implications of the imperfect common knowledge information structure for extracting information from endogenous variables. One exception is the literature the discuss welfare consequences of the public information availability. If public signals become very precise, endogenous variables such as prices and interest rates become less informative about the economic fundamentals. Those variables reflect more information about the public signal instead of fundamentals, which may be welfare decreasing (see Morris and Shin; 2005; Amador and Weill; 2010, 2012; Colombo et al.; 2014; Kohlhas; 2016).

This paper, instead, discuss the implications of the strategic uncertainty induced by the noisy private signal to the learning process from the interest rate. This section highlights differences in learning process between the ICI and ICK models.

ICI model: standard signal extraction

In the imperfect common information (ICI) model, agents observe the whole history of endogenous variables (prices, output and interest rate). Thus, the common information set of firms in period t is given by

$$\mathcal{I}_t = \{i_\tau, \pi_\tau, y_\tau | \tau \leq t\}. \tag{1.20}$$

Firms, however, do not observe all the history of the monetary and inflation target shocks. Since both shocks are within the Taylor rule, inflation and output provide information about those shocks only through their role in the monetary rule. More specifically, firms compute the "adjusted interest rate", $\hat{i}_t \equiv i_t - \phi_\pi \pi_t - \phi_y y_t$, i.e., the interest rate discounted by the endogenous responses to inflation and output. Therefore, all the information content from firms' information set is summarized by the variable $\hat{i}_t$. Then, the firms' observational equation is given by

$$\hat{i}_t = \eta_t - (\phi_\pi - 1)\pi_t^* = C_1 x_t, \tag{1.21}$$

where $C_1 = [1 \quad -(\phi_\pi - 1)]$ and $x_t \equiv [\eta_t \quad \pi_t^*]'$. Therefore, the expectations formation is a standard dynamic signal extraction problem: firms observe a linear combination of shocks but cannot disentangle inflation target changes from monetary shocks. This is the same approach used by Erceg and Levin (2003), Schorfheide (2005), Andolfatto et al. (2008), Del Negro and Eusepi (2011).

Since the model is linear and shocks are normally distributed, the rational expectation is given by the Kalman filter with the observational equation (1.21) and the state equation is given by

$$x_t = A_1 x_{t-1} + \varepsilon_t, \tag{1.22}$$

where $A_1 = diag(\rho_\eta \quad \rho_\pi)$ and $\varepsilon_t = (\varepsilon_t^\eta \quad \varepsilon_t^\pi)$.

The following proposition uses the Kalman filter to write down the expectations formation for the inflation target and monetary shocks.

Proposition 1.1. *Given the observational equation (1.21), the rational expectation about unobservable shocks, x_t, is given by*

$$E_t(x_t) = (I_n - \tilde{K}C_1)A_1 E_{t-1}(x_{t-1}) + \tilde{K}C_1 A_1 x_{t-1} + \tilde{K}C_1\varepsilon_t, \tag{1.23}$$

where $\tilde{K} = \tilde{P}C_1' \left[C_1\tilde{P}C_1'\right]^{-1}$ and $\tilde{P} = A_1(I_n - \tilde{K}C_1)\tilde{P}A_1' + \Sigma_\varepsilon$ are the steady-state Kalman gain matrix and the mean square error of the prediction error of the state, respectively.

The proof is presented in the Appendix 1.A.1. The steady-state Kalman gain matrix $\tilde{K}$ provides the optimal stationary response of the expectation to changes in the observable $\hat{i}_t$. As usual, this matrix depends on the ratio σ_η/σ_π, not on the variances. In this proposition, instead of writing the expectation as a function of the observation, $\hat{i}_t$, I write it as a function of the shocks itself. This form will be helpful to compare to the expectations formation of the ICK model and to solve the model.

ICK model: signal extraction from private information and endogenous variables

There are two additional features in the imperfect common knowledge (ICK) model. First, firms also observe the private signal s_{it} from equation (2.4). Second, I assume the following timing protocol. Every period t is divided into two stages. In stage 1, the inflation target and monetary shocks, and the idiosyncratic signal realize, the central bank sets its interest rate. Firms observe i_t and s_{it}, and set their price $P_{i,t}$ given their information, and (credibly) commit to satisfy the demand at the chosen prices. In stage 2, firms hire labor $L_{i,t}$ to produce and deliver the demanded quantity, $C_{i,t}$. Households also decide their consumption composite C_t, labor supply L_t and demand for one-period nominal bonds, B_t, and markets clear.[14]

[14] In the ICI model, the assumption that agents observe the endogenous variables is equivalent to assume that firms decide prices and quantities simultaneously, which implies that firms can infer the price level and aggregate output via their own price and production decisions. If the timing were the same as in the ICK model, the learning process would be different because firms would not infer y_t by observing y_{it}. Under this

This timing is a standard assumption in the literature (Adam; 2007; Nimark; 2008; Angeletos and La'O; 2009, 2011). Angeletos and La'O (2011) emphasize the importance of this timing in pricing and quantity decisions in the presence of informational frictions, but only in the context of exogenous signals. Here, the timing assumption also affects the signal extraction from the interest rate.

The idiosyncratic signal introduces strategic uncertainty in pricing decisions, which has two key implications for the model. First, the optimal prices of individual firms depend on the aggregate price level. Since firms do not know other firms' pricing decisions, they have to form expectations about this price level. This implies that when aggregating to compute the price level, the price will depend on the average expectation about itself. Therefore, the first-order average expectation of the price will depend on the second-order average, and so on. Then, the ICK Phillips curve (1.14) depends on the hierarchy of expectations about current and future inflation and output. This is emphasized by Nimark (2008) in the case of the New Keynesian Phillips curve, but it is also true for many other environments with strategic complementarities (see Morris and Shin; 2002; Woodford; 2002; Nimark; 2008; Angeletos and La'O; 2009, 2013).

The second implication, which is key for the results, is that the strategic uncertainty affects the capabilities of agents to infer aggregate decisions from their own choices. Specifically, because firms do not known other firms' pricing decisions, they cannot infer the average optimal price $p_t^* = \int_0^1 p_{it}^* di$. Therefore, firms cannot infer the price level from equation (1.12), as they can in the model with imperfect information. Moreover, because firms set prices before production takes place and wages are determined, firms cannot infer the aggregate output. Therefore, the information set of firm $i \in [0,1]$ in period t is given by

$$\mathcal{I}_t^i = \{i_\tau, s_{i\tau}, \pi_{\tau-1}, y_{\tau-1} | \tau \leq t\}. \tag{1.24}$$

That is, firms observe the history of the interest rate and the private signal but only the past history of inflation and output. This has important implications for how firms extract information from the interest rate. To clarify this point, note that the observational equations are the private signal (2.4) and the Taylor rule (3.68). The effect of monetary and inflation target shocks on the interest rate can be decomposed into a direct effect and an endogenous effect such that

alternative timing assumption, the solution method must be modified. The modified method is similar to the approach used in the ICK model described below. Theoretically, this assumption weakens the difference in expectation formation between the ICI and ICK models. In practice, this does not play an important role because interest rates react mostly to inflation, and not to output, i.e., ϕ_y is close to zero, and ϕ_π is greater than one.

$$i_t = \underbrace{\phi_\pi \pi_t + \phi_y y_t}_{\text{Endogenous effect}} + \underbrace{\eta_t - (\phi_\pi - 1)\pi_t^*}_{\text{Direct effect}}. \tag{1.25}$$

In the ICI model, since firms observe π_t and y_t, the endogenous effect is known. Then, firms observe the direct effect, but there is uncertainty about its source. Thus, firms can extract information from the adjusted interest rate shown in equation (1.21), which is by definition the direct effect. In contrast, in the ICK model, there is also uncertainty about the endogenous effect. Interestingly, this results in a learning process that is quite different from the one of the ICI model. For instance, suppose that agents observe an increase in the interest rate. For the ICI model, firms with the information set $\mathcal{I}_t$ observe an increase in $\hat{i}_t$ (positive direct effect), and thus firms expect that it was either a response to a positive monetary shock or a decrease in inflation target (recall that $\phi_\pi > 1$). For the ICK model, a firm with the information set $\mathcal{I}_t^i$ also considers the endogenous effects of inflation and output. Note, however, that higher inflation and output are likely to be responses to a *negative* monetary policy shock or an *increase* in the target. Therefore, the sign of the expectations about shocks is ambiguous, depending on the relative size of the endogenous and direct effects, because they provide information about shocks in opposite directions. Moreover, since the endogenous variables depend on the whole hierarchy of expectations, firms also have to consider the impact of the higher-order expectations of the shocks on the interest rate through their effects on inflation and output. For the same reason, the parameters may affect the expectations formation because they impact the endogenous responses of inflation and output to shocks and their expectations hierarchy, whereas the learning process of the ICI model is affected only by parameters related to the exogenous shocks.

In the following, I present how firms optimally form expectations considering both effects. By the same reasoning as before, the rational expectations can be computed using the Kalman filter, but with different state and observational equations. The state equation is the expectations hierarchy from equation (3.8). The observational equation will reflect the two aforementioned differences. Because π_t and y_t are not observed, firms use their knowledge of the model and the equilibrium interest rate from equation (3.7), and they include the private signal from equation (2.4). Therefore, the observational equation can be written as

$$\begin{bmatrix} i_t \\ s_{it} \end{bmatrix} = \begin{bmatrix} Q_i \\ e_{\pi^*} \end{bmatrix} x_t^{(0:\bar{k})} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_{it} = C x_t^{(0:\bar{k})} + D v_{it}, \tag{1.26}$$

where Q_i is the $1 \times 2(\bar{k}+1)$ subvector of matrix Q from equation (3.7). It describes the equilibrium response of the interest rate to the hierarchy of expectations. e_{π^*} is the selection

matrix such that $\pi_t^* = e_{\pi^*} x_t^{(0:\bar{k})}$, and C and D are defined accordingly. Since the interest rate obeys the Taylor rule, the equilibrium Q_i has to be such that

$$Q_i = \phi_\pi Q_\pi + \phi_y Q_y + C_1 e_x, \tag{1.27}$$

where Q_π and Q_y are the analogous equilibrium responses of inflation and output to the hierarchy of expectations. C_1 is defined in the same manner as in the observation equation of the ICI model (see equation (1.21)), and e_x is the selection matrix such that $x_t = e_x x_t^{(0:\bar{k})}$. Therefore, the learning process is different because it includes both the direct effect and endogenous response of the interest rate to the expectations hierarchy through the the responses of the inflation and output.

The following proposition shows how each firm i's expectation about the hierarchy of expectations is formed, in addition to the average expectation.

Proposition 1.2. *Given the guessed expectation hierarchy (3.8) and the observational equation (1.26), firm i's rational expectation about the expectations hierarchy is given by*

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) A E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB\varepsilon_t + \bar{K}Dv_{it}. \tag{1.28}$$

The average first-order expectation of $x_t^{(0:\bar{k})}$ is such that

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) A E_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB\varepsilon_t, \tag{1.29}$$

where $\bar{K}$ is the steady-state Kalman gain, which can be computed by solving the Riccati equation that combines the following equations:

$$\begin{aligned} \bar{K} &= \bar{P}C'\left[C\bar{P}C' + D\Sigma_v D'\right]^{-1}, \\ \bar{P} &= A\left(I_k - \bar{K}C\right)\bar{P}A' + B\Sigma_\varepsilon B' \end{aligned} \tag{1.30}$$

The proof is presented in Appendix 1.A.2. It also shows in detail how, given the dynamics of first-order expectations from equation (1.29), one can also verify the guessed hierarchy of expectations dynamics from equation (3.8) by finding the equilibrium A and B matrices.

The steady-state Kalman gain matrix $\bar{K}$ provides the optimal stationary response of the expectations hierarchy to changes in the observables i_t and s_{it}. This matrix depends on not only the ratio σ_η/σ_π but also the noise-to-signal ratio σ_v/σ_π.

As in any DSGE model, the equilibrium response of the endogenous variables depends on

the persistence of the shocks. In both the ICI, the responses also depend on the persistence of the expectations about the shocks whereas, in the ICK model, they also depend on the persistence of higher-order average expectations.

The dispersed information model exhibits an interesting interaction between the response of endogenous variables to the expectations hierarchy and expectations formation. Specifically, the endogenous variables depend on expectations about their future, which, in turn, are a function of the future expectation hierarchy. Thus, the response Q depends on the persistence of the higher-order expectations, A. Furthermore, since firms observe the interest rate, the dynamics of the hierarchy of expectations depends on the reaction of the endogenous variables to the hierarchy itself (through the matrix C), i.e., the matrix A depends on Q. Therefore, there is a feedback loop between responses to higher-order expectations and their formation, which is not the case in the imperfect information model since the expectation formation is exogenous in that model.

1.4 Inflation-output trade-off and inflation target expectations

This section presents how the learning about the inflation target affects the trade-off between inflation and output for two policies: a disinflation policy and a contractionary monetary shock. I compare the impulse response functions (IRFs) of inflation and output of the full information (FI), imperfect common information (ICI) and imperfect common knowledge (ICK) models in response to the monetary or inflation target shocks.

The parameters are calibrated using standard values. The discount factor, $\beta = 0.99$, is such that annual interest rate is approximately 4%. The probability of not optimizing prices, $\theta = 2/3$, is set such that firms on average keep prices fixed by 3 quarters. Log utility is assumed ($\gamma = 1$), and the inverse Frisch elasticity is set to be consistent with the macro literature ($\varphi = 2$). Moreover, the elasticity of substitution is chosen such that the firm's mark-up in the steady-state is 11% ($\varepsilon = 10$), following Woodford (2003). The labor share, α, is calibrated as 0.7, consistent with national account data of developed countries. The calibration of those parameters implies weak complementarity in pricing decisions ($\xi = 0.81$). For the monetary policy rule, I use moderate responses to inflation and output ($\phi_\pi = 1.5$, $\phi_y = 0.25$).

The persistence parameters for the inflation target process and monetary shocks are calibrated as $\rho_\pi = 0.975$ and $\rho_m = 0.5$, respectively. Therefore, deviations from the steady-state target are calibrated to be quite persistent, implying a long-lasting change in

monetary policy. In contrast, monetary shocks represent short-lived variation in the interest rate. For instance, the half-life for the former is 27 quarters, whereas that of the latter is 1 quarter. Moreover, for the standard deviations of shocks, I use common estimated values ($\sigma_\eta = 0.15$ and $\sigma_\pi = 0.05$) for medium-scale models, both under perfect information (Adolfson et al.; 2007) and imperfect information (Del Negro and Eusepi; 2011). This calibration implies a ratio $\sigma_\eta/\sigma_\pi = 3$.

The dispersion parameter σ_v is not a standard one and requires more effort to calibrate than the others. From equation (1.28), it is easy to see that v_{it} is the source of heterogeneity of expectations about the hierarchy of expectations. Then, the dispersion in expectations is driven by the variance of the idiosyncratic signal. The unconditional (steady-state) cross-sectional covariance of the expectations hierarchy can then be computed by solving the Lyapunov equation

$$\Sigma = (I_k - \bar{K}C)A\Sigma A'(I_k - \bar{K}C)' + \bar{K}_s\bar{K}_s'\sigma_v^2, \tag{1.31}$$

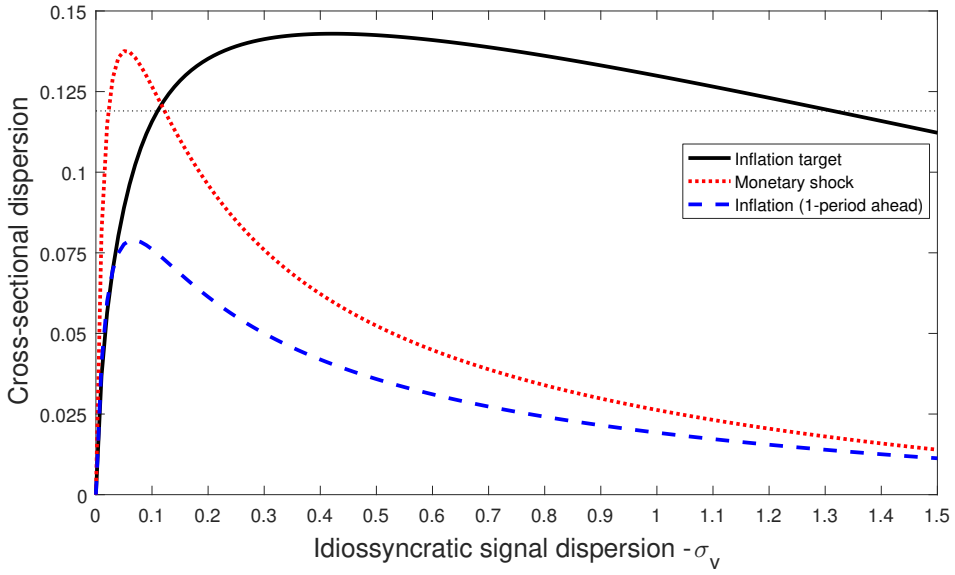
where $\bar{K}_s$ is the steady-state Kalman gain related to the private signal. Formally, $\Sigma \equiv var\left(E_{it}[x_t^{(0:\bar{k})}]\right) = E\left[\left(E_{it}[x_t^{(0:\bar{k})}] - E_t^{(1)}[x_t^{(0:\bar{k})}]\right)\left(E_{it}[x_t^{(0:\bar{k})}] - E_t^{(1)}[x_t^{(0:\bar{k})}]\right)'\right]$. By the definition of x_t, the second element of the diagonal is the cross-sectional variance of the inflation target expectations. Moreover, given the solution of the model from equation (3.7), one can see that the cross-sectional covariance for the endogenous variables can be computed as $Q\Sigma Q'$.

Owing to lack of data regarding expectations about the target, I use as a proxy the longest (10-year ahead) forecast about inflation available from the Survey of Professional Forecasters (SPF). For the period 1991Q1-2015Q4, the quarterly cross-sectional standard deviation is 0.119. It turns out, however, that the relationship between Σ and σ_v is non-monotonic. Note that as σ_v increases, the Kalman gain $\bar{K}_s$ decreases since agents optimally choose to react less to noisier signals. Therefore, the second term in equation (1.31) responds non-monotonically to increases in the private signal dispersion. Figure 1.2 plots the standard deviation of the (first-order) expectation of the inflation target, monetary shock and 1-period ahead inflation as functions of σ_v.[15] The dotted horizontal line refers to the target value of 0.119 from the SPF data.

One can see that two values of σ_v (approximately 0.1 and 1.3) are consistent with the target value. The baseline calibration, $\sigma_v = 0.1$, is chosen for two reasons. First, as Figure 1.2 shows, values of σ_v near 0.1 yield the highest dispersion in expectations about both 1-quarter ahead inflation and monetary shocks. In the SPF data, the cross-sectional standard deviation for the

[15] I show the range values $\sigma_v \in [0, 1.5]$ for expositional convenience. For higher values, as σ_v increases, the cross-sectional dispersion slowly decreases towards zero.

Figure 1.2: Cross-sectional standard deviation as function of σ_v



Note: The dotted horizontal line refers to the target value of 0.119.

1-quarter ahead inflation forecast, for the same period, is approximately 0.20. The model can generate a standard deviation equal to approximately 40% of the one in the data, but it would be unwise to think that only one source of dispersion should be able to fully replicate the disagreement of inflation expectations in the data. Second, the baseline calibration implies a ratio $\sigma_v/\sigma_\pi = 2$, whereas the other option leads to a sufficiently high value that firms almost do not react to the private signal to form expectations about the target.

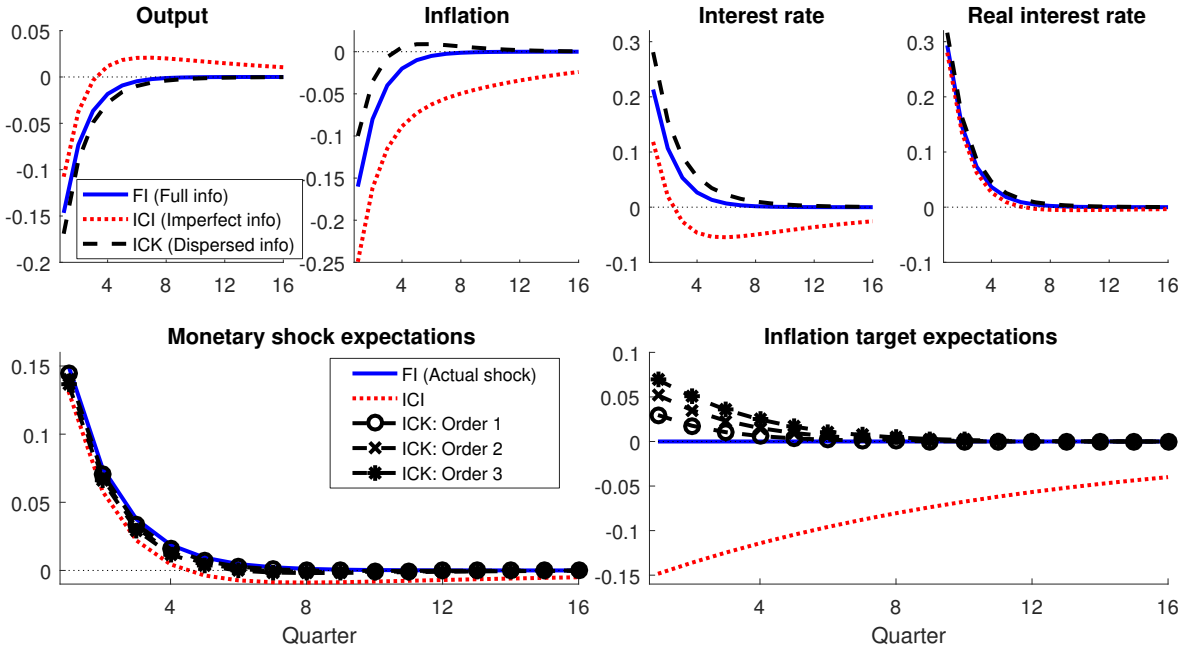
1.4.1 Effects of monetary shocks

This subsection discusses how the differences in expectations formation described in the previous section matter for the effects of monetary shocks. Figure 1.3 shows the IRFs of endogenous variables and the monetary shock and inflation target expectations after a monetary shock for the three aforementioned models. In the bottom panels, the solid blue line shows the actual shock (full information expectation), the dotted red lines show the ICI expectation for both shocks, and the dashed marked black lines show the hierarchy of expectations (up to third order) of the ICK model.[16]

For all models, a positive monetary shock induces lower inflation expectations that decrease inflation and increase the ex-ante real interest rate trajectory, thereby weakening

[16]For the model solution, I use $\bar{k} = 10$, which is more than sufficient for an accurate solution. The graph shows only the first three orders for presentation purposes.

Figure 1.3: Impulse response function after a monetary shock



Note: Response to a one standard-deviation shock. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points.

output. More interestingly, the ICI model has a better trade-off between inflation and output than the full information model – as inflation decreases more with fewer costs in terms of output – whereas the opposite happens for the ICK model.

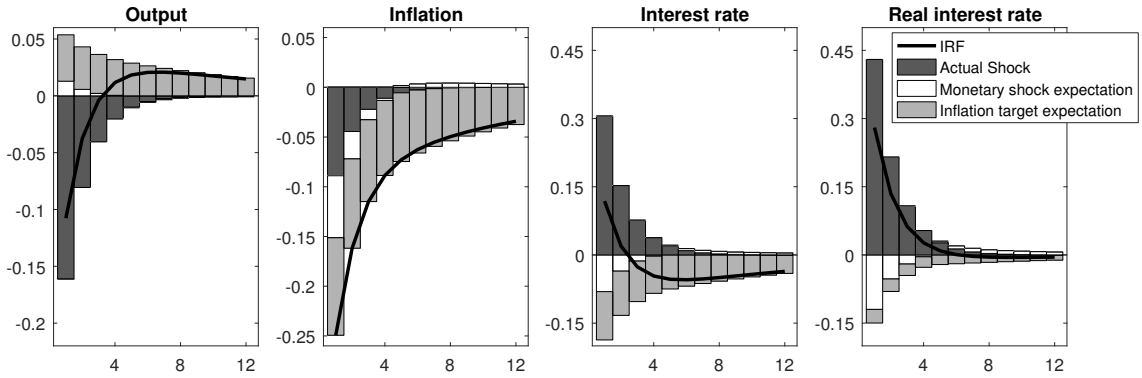
This difference between models in terms of the inflation-output trade-off depend crucially on the differences in the learning process about the inflation target shown in the bottom-right panel. Specifically, for the ICI model, firms beliefs are that inflation target decreased, whereas for the ICK model, the average expectation is that it increased (and average higher-orders are even higher). This implies that inflation expectations – relative to the perfect information benchmark – will be lower for the ICI model and higher for ICK model, thus inducing a relatively higher inflation in the latter. Therefore, in the latter, the central bank sets relatively higher interest rate, leading to higher output costs.

The contrast in the learning process about the inflation target is a reflection of the differences in the signal extraction problem explained in the previous section. That is, the ICK model has to consider the endogenous effects of the hierarchy of expectations about monetary shocks and the inflation target on the interest rate. In other words, firms have to learn about the economy to infer the monetary policy stance. Specifically, firms have to consider whether the interest rate has increased as a response to higher inflation and output,

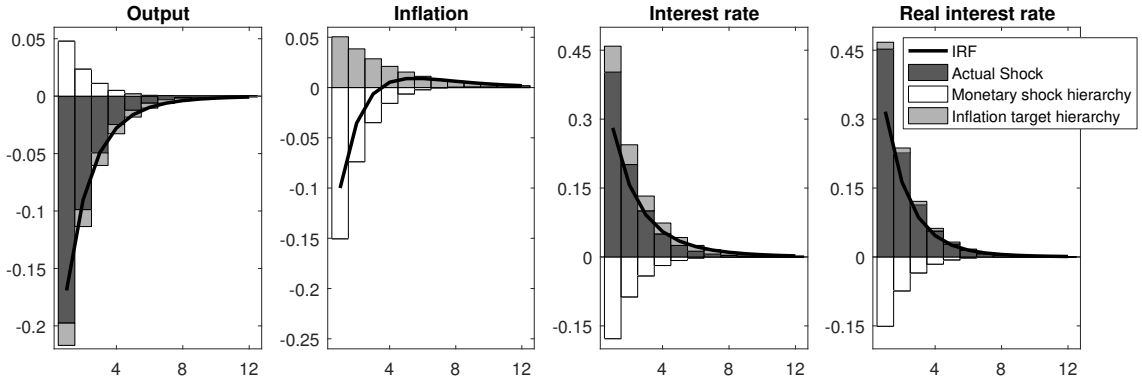
which in turn might be a consequence of a negative monetary shock, or a higher target. For monetary shocks, the endogenous effect is dominant (dominated) when forming expectations about the inflation target (monetary policy).

The intuition is as follows. Because the inflation target is assumed to be a very persistent process, its effect on inflation and output is strong and long-lasting. Then, firms' optimal filtering imply that the increase in the interest rate is more likely to be an outcome of an increase in inflation and output caused by a higher inflation target than a direct effect from a lower inflation target. For monetary shocks, which are much less persistent, the endogenous effect is dominated by the direct effect.

Figure 1.4: IRFs after a monetary shock: expectations hierarchy decomposition of IRFs



(a) ICI model



(b) ICK model

Note: Response to a one standard-deviation monetary shock. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points. Appendix 1.A.4 shows how to compute this decomposition.

Figure 1.4 emphasizes the impacts of the expectation formation differences on each endogenous variable. Specifically, the IRFs (black solid lines) are decomposed into the contribution of the monetary shock, its expectations, and the inflation target expectations

(and all the hierarchy of expectations for the ICK model). Panel (a) shows the results for the ICI model, and panel (b) shows those for the ICK model. Appendix 1.A.4 discusses how to compute this decomposition based on the model solution and the expectations dynamics.

In the ICI model, as shown by the dark gray bar, a monetary shock increases the interest rate and decreases inflation expectations, leading to lower inflation and a higher real interest rate (since $\phi_\pi > 1$). This implies a decrease in output, which reinforces the decrease in inflation. Moreover, firms expect a positive monetary shock and a lower inflation target (recall from Figure 1.3). The white and light gray bars show that expectations about both shocks intensify the decrease in inflation. This has a negative impact on both the nominal and real interest rate through the endogenous part of the Taylor rule, thereby leading to a positive impact on output according to the Euler equation. The reinforcing effect on inflation is sufficiently strong to push the interest rate below the steady-state value in the third quarter after the shock.

Panel (b) shows two important differences between the ICK and ICI models. First, whereas the effects of the actual shock and its (higher-order) expectations have similar effects as their counterparts in the ICI model, one qualification is important. Since output is not observable by the firms in the ICK model, only expectations about output matter, instead of output itself (compare equations (1.14) and (1.15)). Therefore, the actual shock has no impact on inflation, and the decrease in inflation is caused only by expectations about a positive monetary shock. Second and more interestingly, the inflation target expectations have the opposite effect than in the ICI model. Since firms, on average, expect a higher inflation target, this expectation counterbalances the negative effect of a higher interest rate on inflation through relatively higher inflation expectations. This leads the central bank to increase even further the interest rate owing to the endogenous effect of inflation, which reinforces the decrease in output.

In summary, firms in the ICK model expect that a higher inflation target, leading to a worse inflation-output trade-off than in the full information model since inflation expectations are relatively higher. The opposite is true for the ICI model: the trade-off is better than in the full information model because inflation expectations are lower.

1.4.2 Disinflation period: a lower inflation target

Now I turn to the relation of disinflation costs and the learning process of the target. Figure 1.5 shows the IRFs for a decrease in the inflation target. Note that the inflation target is highly persistent, but the process is still stationary; thus, all variables will eventually return to their steady-state values. More importantly, as emphasized by Mankiw and Reis (2002), Erceg and Levin (2003) and others, the full information model is unable to present realistic

costs in terms of output for disinflations.[17] In this setting, the decrease in inflation target is known by firms, who adjust their inflation expectations accordingly, which implies that real interest rate exhibits a negligible increase that leads to essentially no output costs.

Both the ICI and ICK models can account for the costs in terms of output of the disinflation because firms have to learn that the inflation target has decreased. Therefore, inflation expectations do not decrease as much as in the full information model, implying relatively higher real interest rates leading to a decrease in output. For the baseline calibration, it is clear that there is a worse trade-off between inflation and output in the ICI model than in the ICK model, as inflation is relatively higher and output relatively lower for almost all the time after the change in the target. This is the case because the learning process about the target is much faster in the ICK model, as shown in the bottom-right panel of Figure 1.5. This depends crucially on the fact that σ_v/σ_π is relatively small, consistent with the data regarding long-term inflation expectations. Intuitively, the costs of disinflation arise from the fact that real interest rates are relatively higher since inflation expectations slowly incorporate the fact that the inflation target decreased. Then, smaller σ_v/σ_π corresponds to better-informed firms and lower output costs.

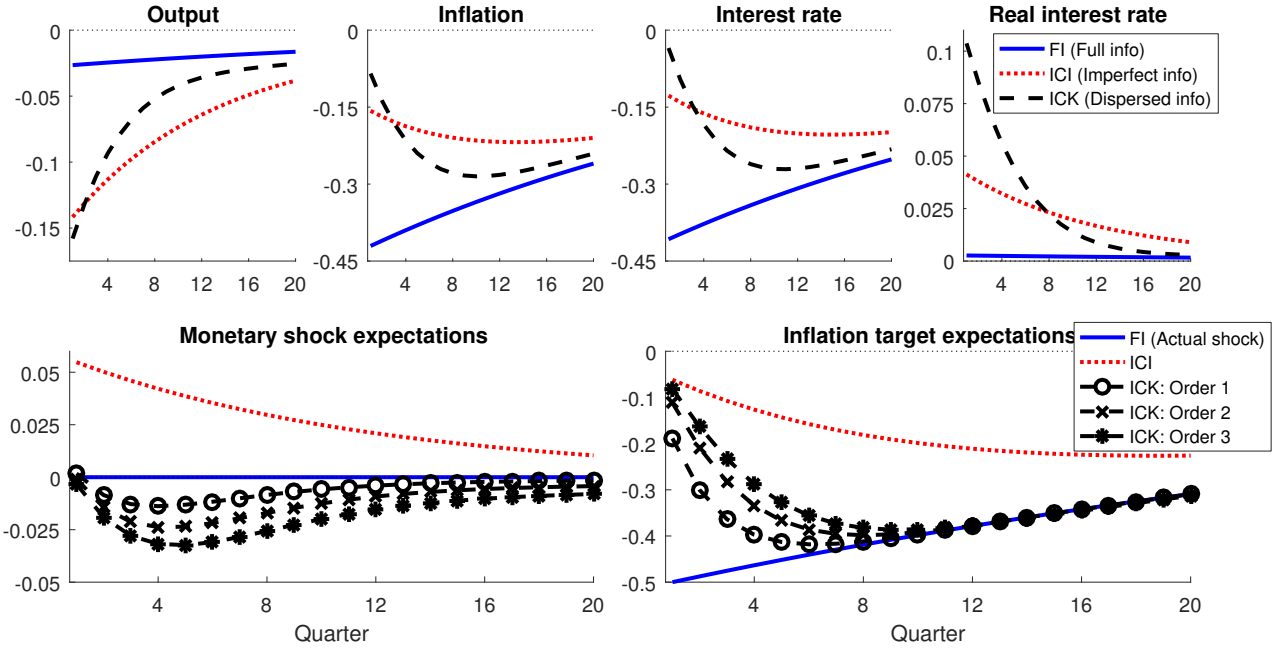
Moreover, for the target shock, the learning about the inflation target are similar whereas the qualitative differences are in the learning process are for the monetary shock (see bottom panels of Figure 1.5). This difference is the opposite of the one from the monetary shock shown above.

The reason is the following. For the monetary shock, both interest rate and the adjusted interest rate increase after the shock. For the (negative) target shock, the endogenous effect is strong enough to drive the interest rate below steady-state for both models (see Figure 1.5). Despite the decrease in the interest rate, the adjusted interest rate, $\hat{i}_t$, increases (see equation (1.21)). Then, ICI and ICK model extract information from variables that go in different directions – the opposite situation from the monetary shock –, which implies a switch in which shocks the qualitative differences in learning happens.

In other words, for the ICI model, firms expect that the increase in the adjusted interest rate is a result of a lower inflation target or a positive monetary shock. For the ICK model, the direct effect is dominant for monetary shocks, and the endogenous effect dominates for the inflation target. For the latter, firms find it more likely that the interest decrease is an endogenous response of inflation and output to a lower inflation target than a direct effect of a higher target. The opposite happens for the expectations of monetary shocks, except for the initial impact, since the initial decrease in the interest rate is relatively small. Firms find

[17] Ball (1994b) provide evidence of costs of disinflation for a panel of countries.

Figure 1.5: Impulse response function after a decrease in the inflation target



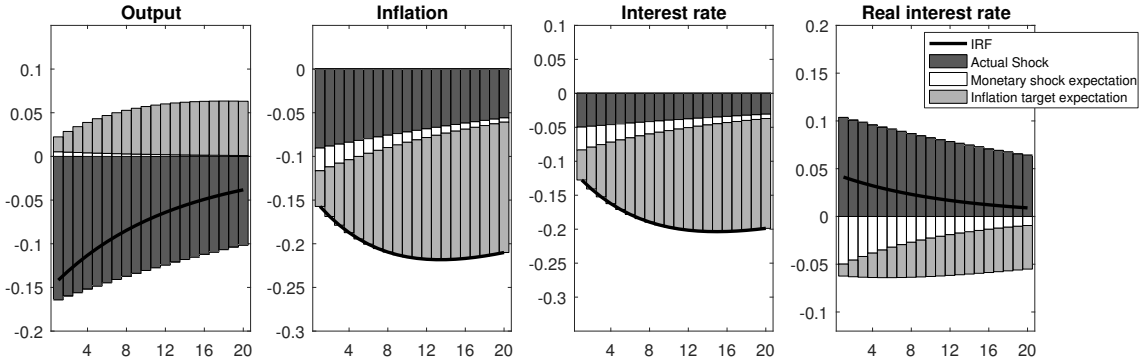
Note: Response to a 0.5 p.p. decrease in the inflation target. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points.

it more likely that the decrease in interest rate is a direct effect of a negative monetary shock than an endogenous decrease in output and inflation owing to a positive monetary shock.

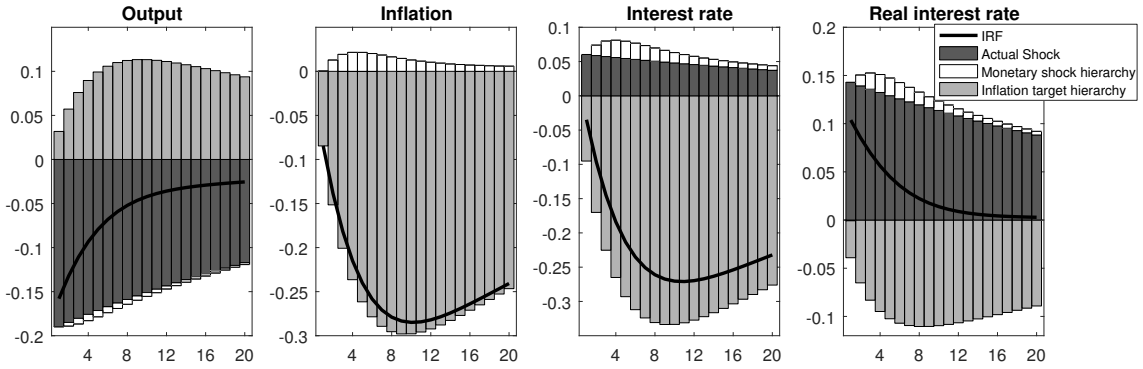
In contrast with monetary shocks, the difference in learning does not play an important role for the target shock. The expectation hierarchy decomposition of the IRFs from Figure 1.6 illustrate the reason for this. In panel (a) it is clear that the expectation about a positive monetary shock (white bar) induces both lower inflation and real interest rate that leads to a small positive effect on output. In the panel (b), firms on average expect a negative monetary shock that has a positive effect on inflation and the real interest rate (since $\phi_\pi > 1$). Therefore, whereas in the former, expectations about the monetary shock reinforce the effect of the shock, they do the opposite in the latter. However, those effects are rather small to counterbalance the effects of the decrease in the inflation target. This is the case because the inflation target shock is very persistent, whereas monetary shocks are short-lived.

The models deliver a counterfactual empirical prediction that in disinflations, the nominal interest rate should decrease (although both correctly capture the increase in real rates). The ICK model performs relatively better than the ICI model in this sense because the interest rate almost does not change on impact and gradually decreases after a target shock. In imperfect information models with richer dynamics, such as that of Erceg and Levin (2003),

Figure 1.6: IRFs after a decrease in inflation target: expectations hierarchy decomposition



(a) ICI model



(b) ICK model

Note: Response to a decrease of 0.5 p.p. in the inflation target. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points. Appendix 1.A.4 shows how to compute this decomposition.

this problem is corrected, which might be also true for the ICK model.

1.4.3 Strategic uncertainty and learning from endogenous variables

The introduction of the noisy signal, s_{it}, affects the signal extraction in two aspects. It provides additional information about the inflation target but also prevents firms from observing the current inflation and output. Previously, I argued that the key difference between the models is the fact that firms need to consider the endogenous effect of shocks when forming expectations. In this section, I provide two different evaluations. First, a sensitivity analysis of the results is performed to check whether the difference in the learning process about the target remains when key parameters are perturbed. Second, by imposing a counterfactual information set to firms, I demonstrate that the main difference in the learning is not a result of the additional information that the signal s_{it} itself provides but rather the

fact that firms cannot infer the output and inflation from their own decisions.

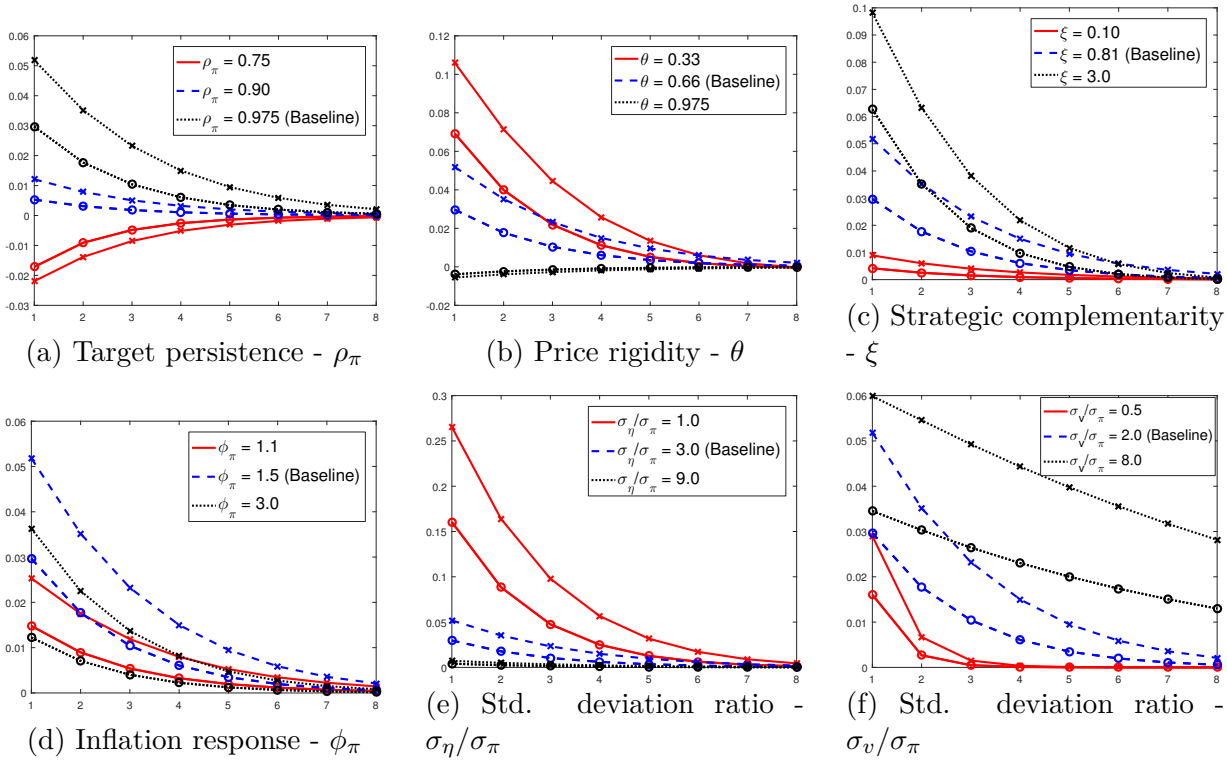
Figure 1.7 shows the IRFs of the first-order (dashed lines with "$\circ$" marker) and second-order (dashed lines with "$\times$" marker) expectations about the inflation target after a monetary shock. Each panel shows the counterpart of the bottom-right plot in Figure 1.3 for different parameters. All plots show the IRFs under three different calibrations for each parameter, keeping all other parameters fixed to their baseline values.

Plot (a) shows that the expectations about the inflation target are positive as long as the persistence about the target is sufficiently high. Higher persistence of the target implies a higher effect of expectations about the target on inflation and output, i.e., a more pronounced endogenous effect on the Taylor rule. This increases the likelihood that the rise in the interest rate was due to an higher target. Plot (b) shows that for most of the calibrations, firms expect that the inflation target increases after a monetary shock. Only when prices are extremely rigid ($\theta = 0.975$, which implies that prices are fixed, on average, for 40 quarters) this is not the case. Intuitively, the more rigid prices are, the flatter the Phillips curve becomes, implying weaker effects of the shocks to the interest rate on inflation. Therefore, firms find it less likely that the interest rate increased as a result of higher inflation. For all other parameters shown in Figure 1.7, the expectations are consistent with the baseline result, even for extreme calibrated values. Moreover, they are all consistent with the endogenous part of the interest rate rule leading to expectations of an increase in the target. Panel (c) shows that both strong strategic complementarities ($\xi = 0.1$) or substitutability ($\xi = 3$) in pricing decisions are consistent with expectations of an increase in the target. Stronger strategic complementarity in pricing decisions is associated with higher expectations about the target since the Phillips curve becomes flatter.

Panel (d) shows that the parameter that controls the response of the interest rate to inflation has an interesting non-monotonic effect on target expectations. For relatively low values of ϕ_π, a higher response leads to higher inflation target expectations (compare the red lines with the blue ones). Instead, for relatively high values, a higher response leads to lower target expectations (compare the blue lines with the black ones). Intuitively, a higher ϕ_π increases the endogenous effect because it amplifies the response of the interest rate to inflation, but it also decreases the effects of both shocks on inflation because firms' expectations take into account that the central bank is more aggressive towards inflation. The second effect dominates as the value of ϕ_π become high.

The last two parameters shown are the noise-to-signal ratios. An higher ratio σ_η/σ_π (panel (e)) implies a smaller increase in expectation about the target. In words, when the monetary shock is relatively noisier than the target shock, the interest rate provides

Figure 1.7: Sensitivity analysis: IRFs of inflation target expectations after a monetary shock



(a) Target persistence - ρ_π (b) Price rigidity - θ (c) Strategic complementarity - ξ

(d) Inflation response - ϕ_π (e) Std. deviation ratio - σ_η/σ_π (f) Std. deviation ratio - σ_v/σ_π

Note: Markers "$\circ$" and "$\times$" denote average first-order and second-order expectations, respectively. Responses to a one standard-deviation monetary shock. Inflation target expectations are annualized and expressed in percentage points.

less reliable information. Thus, firms rely relatively more on their private signal, which, on average, indicates that the target is unchanged. Panel (f) shows that the higher noise-to-signal ratio, σ_v/σ_π, the more pronounced is the increase in the inflation target expectations . This is the case since, on average, the information from the private signal is that the target is unchanged, whereas the information from the interest rate is that the target is increased. Therefore, the noisier the information from the private signal, the more strongly firms will weight the information from the interest rate and thus on average expect a higher target.

In summary, the results shown in Figure 1.7 reinforce that the key reason for the differences in the expectations formation is the high persistence of the target. Even extreme deviations of other parameters from the baseline calibration still sustain the qualitative learning differences about the inflation target after a monetary shock.[18] It is worth noting that only the persistence

[18]Evidently, one can pick a combination of parameters with an extreme calibration that attenuates the endogenous forces of the monetary rule leading to a similar learning process between the ICK and ICI models.

parameters (ρ_π and ρ_η) and the noise-to-signal ratio (σ_η/σ_π) affect the learning process of the ICI model, as in any standard signal extraction problem. In the ICK model, however, all parameters affect the learning process because they influence the response of the interest rate to shocks and their hierarchy of expectations through the endogenous part of the rule.

Now I turn to the relative importance of the additional information about the target from the private signal vis-a-vis the lack of information about inflation and output induced by the strategic uncertainty to explain the results. To do so, consider the following thought experiment. Impose that firms in the ICK model observe the adjusted interest rate $\hat{i}_t$ instead of i_t. Therefore, the information set of each firm i given by

$$\hat{\mathcal{I}}_t^i = \{\hat{i}_\tau, s_{i\tau}, \pi_{\tau-1}, y_{\tau-1} | \tau \geq t\}, \tag{1.32}$$

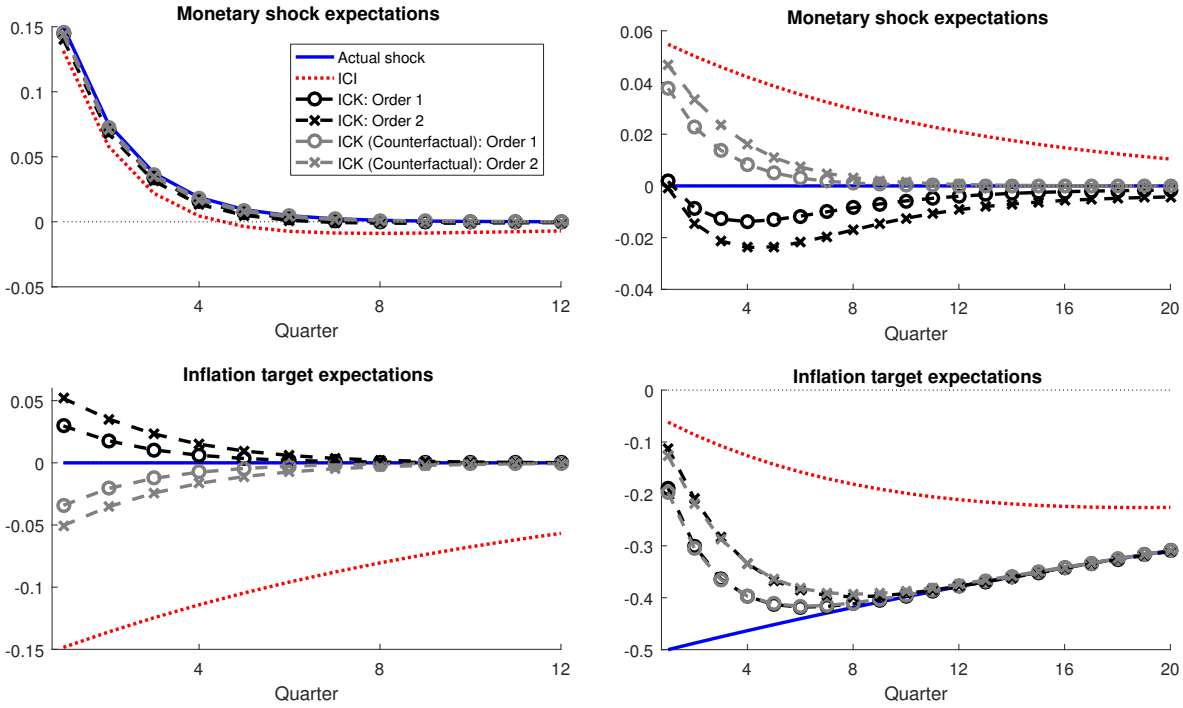
instead of the one in equation (1.24). I refer to $\hat{\mathcal{I}}_t^i$ as the counterfactual information set in the following. Note that this assumption is not the same of imposing that firms observe inflation and output since, in that case, it would affect the Phillips curve (1.14).[19] In other words, firms have additional information from the private signal without losing any information from the interest rate. In such case, the state equation is the expectations hierarchy from equation (3.8), and the observation equations are equations (1.21) and (2.4).

Figure 1.8 compares the IRFs of the expectations about the shocks under the counterfactual information set with the true information sets after a monetary shock (panel (a)) and after a decrease in the inflation target (panel (b)). In both panels, it is clear that the qualitative learning difference vanishes when one considers the counterfactual information set for the ICK model. Under this modified information set, firms expectations respond in a similar manner as in the ICI model but are more accurate. Moreover, the additional information from the private signal does not help to explain the qualitative differences in learning about the shocks – rather, the opposite is true. For the monetary shock, the learning differences are in terms of the inflation target. The private signal, on average, indicates that the target did not change and thus implies *lower* inflation target expectations for the ICK (recall the discussion of panel (f) of Figure 1.7), unlike the benchmark case. Moreover, for the target shock, the public signal indicates a target decrease, implying that firms find it less likely that the interest rate decreased because of a negative monetary shock. This goes in the opposite direction of explaining the expectations about the target in the baseline ICK model.

Therefore, the key mechanism through which expectations differ is the change in capability to extract information from the interest rate induced by the strategic uncertainty, not the additional information about the target from the private signal.

[19] Imposing this stronger version leads to similar results.

Figure 1.8: Counterfactual information set: IRFs of monetary shock and inflation target expectations



(a) Monetary shock (b) Inflation target shock

Note: Panel (a) shows the responses to a one standard-deviation monetary shock. Panel (b) shows the responses to a 0.5 p.p. decrease in the inflation target. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points.

1.5 Transparency and the effectiveness of monetary policy

This section discusses how transparency regarding the inflation target can affect the effectiveness of the monetary policy and demonstrates the importance of those learning differences in this particular context. Suppose that firms also observe a noisy public signal about the inflation target given by

$$s_t^p = \pi_t^* + u_t, \tag{1.33}$$

where $u_t \sim \mathcal{N}(0, \sigma_u^2)$. One interpretation of this public signal is that it represents the communication from central bank about its goals. In that case, σ_u is a measure of the degree of (lack of) transparency. Following Faust and Svensson (2002), if $\sigma_u = 0$, firms observe the true inflation target. In that case, the central bank is fully transparent and both ICI and ICK

models are equivalent to the full information model. For a sufficiently high σ_u ($\sigma_u \to \infty$), the equilibrium with the public signal is exactly the same as one without the signal because it provides no useful information about the target. In that case, the central bank is fully opaque. Finally, for any intermediate value of σ_u, the model is subject to imperfect transparency.

This modeling device is a simple and reduced form to define transparency in terms of the accuracy of the communication of the central bank about its actual target π_t^* (recall that the steady-state target Π^* is common knowledge). An alternative interpretation is that σ_u controls the confidence bands around the inflation target, which is related to the common practice of central banks under inflation-targeting regimes. Of course, in the model, the public signal is centered around the time-varying inflation target, whereas in practice, the bands are defined around the announced fixed inflation target. On the other hand, we cannot know for sure whether the central bank is aiming at any point within the band or the announced target itself.

To calibrate this parameter, I choose $\sigma_u/\sigma_\pi = 2.5$. This implies that the public information is less informative than the private information. This calibration (equivalent to $\sigma_u = 0.125$) corresponds to an approximately 1.0 annual percentage (95% confidence) band. However, the results are qualitative and hold for any value of σ_u that provides meaningful additional information for the firms.

The additional information is straightforward to implement in the ICI model because one simply has to include the public signal in the observational equation (1.21), and the solution method is the same. For the ICK model, however, the solution method requires two important modifications. First, the higher-order expectations will also depend on the noise of the public signal. Second, when forming individual expectations, firms have to take into account that the public signal affects the average expectation.

Formally, the guessed dynamics for the higher-order expectations is such that

$$x_t^{(0:\bar{k})} = A x_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t. \tag{1.34}$$

Moreover, by adding the public signal to the observational equation (1.26), the observational equation can be written as

$$\begin{bmatrix} i_t \\ s_{it} \\ s_t^p \end{bmatrix} = \begin{bmatrix} Q_i \\ e_{\pi^*} \\ e_{\pi^*} \end{bmatrix} x_t^{(0:\bar{k})} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_{it} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_t = C x_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t, \tag{1.35}$$

where Q_i and e_{π^*} are defined as before and C, D_v and D_u are defined accordingly.

The following proposition shows how expectations about the hierarchy of expectations are formed. It extends the Proposition 1.2 by including public signals as observables.

Proposition 1.3. *Given the guessed expectation hierarchy (1.34) and the observational equation (1.35), firm i's rational expectation about the expectations hierarchy is given by*

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}\left(CB_u + D_u\right)u_t + \bar{K}D_v v_{it}. \tag{1.36}$$

The average first-order expectation of $x_t^{(0:\bar{k})}$ is such that

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) AE_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}\left(CB_u + D_u\right)u_t \tag{1.37}$$

where $\bar{K}$ is the steady-state Kalman gain, which can be computed by solving the Riccati equation that combines the following equations:

$$\begin{aligned} \bar{P} &= A\left[\bar{P} - \bar{K}\left(\bar{P}C' + \Sigma_{cov}\right)'\right]A' + \Sigma_{state}, \\ \bar{K} &= \left[\bar{P}C' + \Sigma_{cov}\right]\left[C\bar{P}C' + \Sigma_{obs}\right]^{-1}, \end{aligned} \tag{1.38}$$

where $\Sigma_{obs} = D_v\Sigma_v D_v' + D_u\Sigma_u D_u'$, $\Sigma_{state} = B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u'$ and $\Sigma_{cov} = B_u\Sigma_u D_u'$.

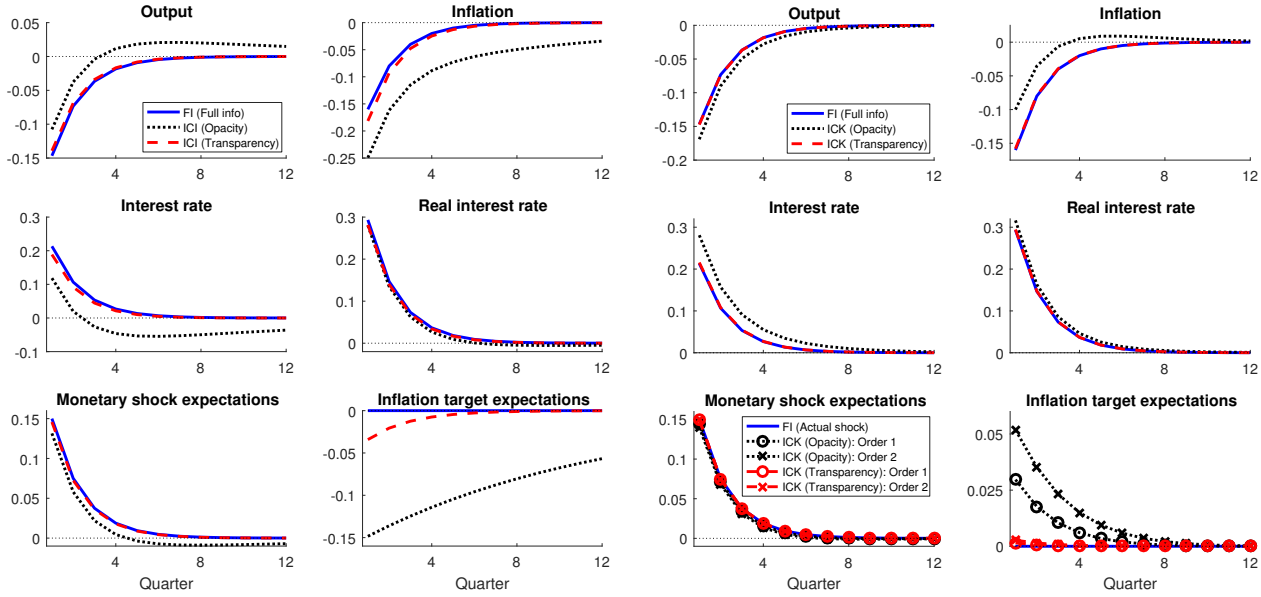
The proof is in the Appendix 1.A.2. It also shows in detail how, given the dynamics of first-order expectations from equation (1.37), one can also verify the guessed hierarchy of expectations dynamics from equation (3.8) by finding the equilibrium A, B_ε and B_u matrices.

Note that whereas the idiosyncratic noise v_{it} is washed-out in the average expectations, the noise of the public signal, u_t, remains. Therefore, u_t is an additional source of variation for endogenous variables but only through changes in average expectations. For instance, Lorenzoni (2009), and Angeletos and La'O (2010) discuss the importance of those noise shocks about productivity to explain business cycles fluctuations.[20]

When using the Kalman filter, firms have to take into account the correlation of errors of the state and observational equations (u_t appears in both equations). This is why the term Σ_{cov} appears in the equation (1.38). Intuitively, when forming expectations about higher-order average expectations, agents take into account that public signals also affect other agents' expectations. This result relates to Morris and Shin (2002), which show that agents with

[20] One difference from those papers is that I do not impose the simplifying assumption that period $t-1$ shocks become common knowledge at period t.

Figure 1.9: Transparency and inflation-output trade-off: IRFs after a monetary shock



(a) ICI model (b) ICK model

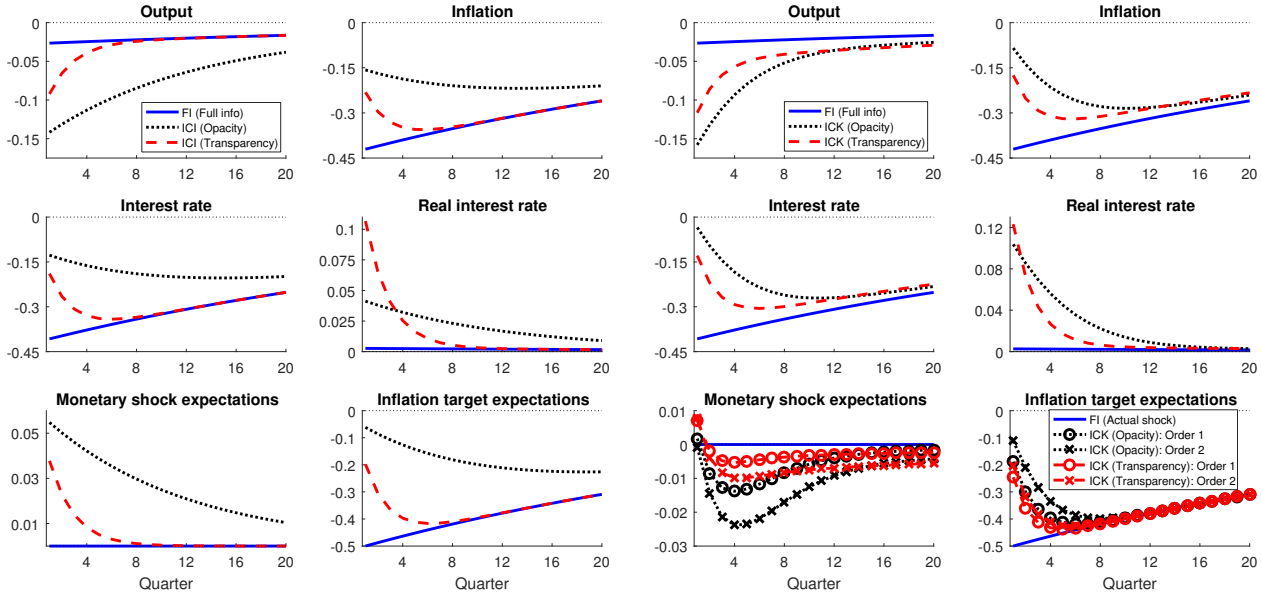
Note: Response to a one standard-deviation shock. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points.

strategic complementarities tend to overreact to public signals vis-a-vis private signals relative to the reaction expected by their relative precision.

Figure 1.9 compares the IRFs after a monetary shock under imperfect transparency (with a public signal) and under opacity (no public signal). Panels (a) and (b) show the IRFs for the ICI and ICK models, respectively. For both models, one can see that the additional information from the public signal is sufficiently precise to induce more accuracy of firms' expectations and faster learning towards the true values of the shocks. How fast firms learn about the inflation target depends on σ_η/σ_π and σ_u/σ_π for the ICI model and also on σ_v/σ_π for the ICK model.

In summary, both models become closer to the full information model as the public information about the target becomes more accurate. For the ICI model, this implies *higher* inflation expectations than without the public signal, since firms expect a lower inflation target than otherwise. Then, the central bank responds with stricter tightening in the interest rate, leading to a sharper decrease in output with higher inflation. In that case, higher transparency would lead to a *worse* inflation-output trade-off. In contrast, in the ICK model, firms expect on average that the inflation target increased. The information from the public signal implies that firms lower their expectations about the target. This results in

Figure 1.10: Transparency and inflation-output trade-offs under disinflation



(a) ICI model (b) ICK model

Note: Response to a 0.5 p.p. decrease in the inflation target. Inflation, interest rates and inflation target expectations are annualized and expressed in percentage points.

lower inflation expectations than otherwise and thus lower inflation with lower output costs. Therefore, higher transparency implies a *better* inflation-output trade-off. Thus, differences in the expectations formation play an important role in assessments of the effects of transparency on the effectiveness of the monetary policy.

Figure 1.10 shows the same plot for the case of a decrease of 0.5% in the inflation target. For both the ICI and ICK models, higher transparency diminishes uncertainty about the inflation target such that expectations about the target are lower and converge faster to the true value of the target than otherwise. For the baseline calibration, the effect of transparency is more pronounced in the ICI model since it takes longer for firms to learn about the target in this model. For both models, because of the forwarding-looking nature of the Euler equation, output costs are lower under transparency – despite the initially higher real interest rate – since the accumulated real interest rate during the whole period is lower. Therefore, a more transparent monetary policy in terms of the inflation target results in less costly disinflation.

1.5.1 Discussion

By adding a public noisy signal to a standard imperfect information model that the recent literature used for disinflation periods, I study the effects of transparency on the effectiveness

of monetary policy. Whereas for disinflation periods, more transparency implies lower costs in terms of output, this is not the case for monetary shocks. This difference is particularly troublesome since the focus on transparency exhibited by central banks is not restricted to central banks that are pursuing disinflation policies. Rather, this trend toward more transparency is a widespread feature of modern central banking.

Surprisingly, the effect of higher transparency on the trade-off between inflation and output depends on the details of the learning process, even with a small change in the information available to firms – i.e., the introduction of a private noisy signal about the target. For the disinflation case, this modification affects the results only quantitatively, but it changes the results qualitatively for the monetary shock case. Specifically, it changes the conclusions about the relationship between the transparency about the target and the effectiveness of monetary policy. For the dispersed information model, irrespective of which type of shock is considered, more transparency results in a lower inflation-output trade-off.

A priori, one would think that the introduction of a noisy idiosyncratic signal should not be key to answer the question on which I focus here. However, the private signal not only provides an additional information about the target but also changes the nature of the learning process from the interest rate. This change leads to a richer and more realistic learning process about the inflation target and the monetary policy shock. The uncertainty about monetary policy that firms face translates into uncertainty about the economy, which, in turn, introduces a confounding factor into the interest rate. This affects formation of expectations about monetary policy. This type of consideration seems consistent with the fact that central banks communicate to explain the reasons for their interest rate decisions, which are usually related to developments in inflation and economic activity. I believe that this feedback interaction about learning about the economy and learning the reasons for a change in the interest rate is a better representation of the challenges that forecasters face than the one from the standard signal extraction problem based on the imperfect information model. In other words, the standard imperfect information model imposes that agents have too much information about the state of the economy. This matters when they are extracting information from endogenous variables since the fact that they know that other agents' decisions are exactly like theirs might provide some useful information about those variables.

1.6 Concluding remarks

In this paper, I study whether a central bank's transparency about its goals increases the effectiveness of the monetary policy in terms of the inflation-output trade-off. If the

widespread view of central banks that transparency is an important tool to control inflation with lower output costs is correct, then the imperfect common knowledge model provides a better explanation of the relationship between transparency and the effectiveness of monetary policy. In order study transparency within the imperfect common knowledge framework, this paper also builds on Nimark (2008) and Melosi (2017) to extend the solution method of the imperfect common knowledge model to include exogenous public signals.

Strategic uncertainty induced by the noisy idiosyncratic signal generates interesting and richer learning about the monetary policy stance. Agents must form expectations about economic developments to understand the reasons for the monetary authority's decisions, whereas the imperfect information model has a mechanical learning process from an exogenous signal extraction problem. I argue that the former provides a better description of the real uncertainty that agents face when a central bank changes the interest rate.

This paper abstracts from the uncertainty about real shocks that might interact with the expectations formation about the monetary shocks, as discussed by Melosi (2017). Furthermore, the reasons for the inflation target change, and the optimal monetary and communication policies are not explored. Those are interesting and challenging avenues for future research.

1.A Proofs

1.A.1 Proof of Proposition 1.1

Using the Kalman filter with the state equation as (1.22) and observational equation (1.21), one can write the update equation as

$$E_t(x_t) = E_{t-1}(x_t) + K_t[\hat{i}_t - E_{t-1}(\hat{i}_t)], \tag{1.A.1}$$

where the Kalman gain, K_t, and mean square error of the one-step ahead prediction error, $P_{t+1|t}$, have the following dynamics:

$$\begin{aligned} K_t &= P_{t|t-1}C_1' \left[C_1 P_{t|t-1} C_1'\right]^{-1} \\ P_{t+1|t} &= A_1(I_n - K_t C_1)P_{t|t-1}A_1' + \Sigma_\varepsilon. \end{aligned} \tag{1.A.2}$$

Since x_t is stationary and Σ_ε is positive definite, then there exists a steady-state solution such that $\tilde{P} = P_{t+1|t} = P_{t|t-1}$ which implies the steady-state Kalman gain $\tilde{K} = \tilde{P}C_1' \left[C_1 \tilde{P} C_1'\right]^{-1}$ (see Hamilton; 1995, chap. 13). Therefore, taking expectations of $\hat{i}_t$ and using the fact that $E_{t-1}(x_t) = A_1 E_{t-1}(x_{t-1})$, the dynamics of $E_t(x_t)$ can be written as equation as

$$E_t(x_t) = (I_n - \tilde{K}C_1)E_{t-1}(x_{t-1}) + \tilde{K}\hat{i}_t. \tag{1.A.3}$$

Moreover, substituting (1.21) into the equation above and using equation (1.22) one can find the equation (1.23) in the Proposition 1.1.

In the more general setup, when the public signal is included in the information set of the firms, the observation equation becomes

$$\begin{bmatrix} \hat{i}_t \\ s_t^p \end{bmatrix} = \begin{bmatrix} 1 & -(\phi_\pi - 1) \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t = C_1 x_t + D u_t, \tag{1.A.4}$$

where C_1 and D are defined accordingly. Following the same steps

$$E_t(x_t) = (I_n - \tilde{K}C_1)A_1E_{t-1}(x_{t-1}) + \tilde{K}C_1A_1x_{t-1} + \tilde{K}C_1\varepsilon_t + \tilde{K}Du_t, \tag{1.A.5}$$

where $\tilde{K} = \tilde{P}C_1' \left[C_1\tilde{P}C_1'\right]^{-1} + D\Sigma_u D'$ and $\tilde{P} = A_1(I_n - \tilde{K}C_1)\tilde{P}A_1' + \Sigma_\varepsilon$ are the steady-state Kalman gain matrix and the mean square error of the prediction error of the state, respectively. □

1.A.2 Proof of Propositions 1.2 and 1.3

In this subsection, I show the proof of the Proposition 1.3 and discuss that the results of Proposition 1.2 is a special case when there is no public signals in agents' information set.

Consider firm i with the observational equation (1.35), restated by convenience:

$$Z_{it} = Cx_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t, \tag{1.A.6}$$

where $Z_{it} = [\ i_t \quad s_{it} \quad s_t^p\ ]'$ and C, D_v and D_u are defined as before. In general, if agents observe endogenous variables, is a C linear transformation of Q, such as $C = c_0 + c_1 Q$, where c_0 and c_1 are conformable matrices. The proof here works for any observational equation such that Z_{it} is a $q \times 1$ vector, C is a $q \times k$ matrix, D_v is a $q \times p_v$ matrix and D_u is a $q \times p_u$ matrix. Note that if $D_u = 0_{q \times p_u}$, this equation is in the same form as the baseline observation equation (1.26). Moreover, firms are forming expectations about the hierarchy of expectations that is given by the state equation (1.34), also restated:

$$x_t^{(0:\bar{k})} = Ax_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t. \tag{1.A.7}$$

In the following, I show that if $D_u = 0_{q \times p_u}$, then $B_u = 0_{k \times p_u}$, i.e., the state equation becomes the same as in the baseline case (equation (1.34)). Therefore, this proof works for

both Propositions 1.2 and 1.3.

Proof. Each firm i uses the Kalman filter and find the update equation given by

$$E_{i,t}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Z_{it} - E_{i,t-1}\left[Z_{it}\right]\right], \tag{1.A.8}$$

where K_t is the Kalman gain given by

$$K_t = \left[P_{t/t-1}C' + \Sigma_{cov}\right]\left[CP_{t/t-1}C' + \Sigma_{obs}\right]^{-1}, \tag{1.A.9}$$

where $\Sigma_{obs} = D_v\Sigma_v D_v' + D_u\Sigma_u D_u'$ and and $\Sigma_{cov} = B_u\Sigma_u D_u'$. As usual, the mean squared error (MSE) of the one-period ahead prediction error is given by

$$P_{t+1/t} = A\left[P_{t/t-1} - K_t\left[CP_{t/t-1} + \Sigma_{cov}\right]\right]A' + \Sigma_{state}, \tag{1.A.10}$$

where $\Sigma_{state} = B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u'$. For details of this deviation, see for instance Hamilton (1995, chap. 13).

Using the observational equation, (1.A.6), taking expectations and inserting in (3.A.3) one can find:

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Cx_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t - CE_{i,t-1}\left[x_t^{(0:\bar{k})}\right]\right] \tag{1.A.11}$$

Therefore, one can rewrite the equation above as

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - K_tC\right)E_{it-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Cx_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t\right] \tag{1.A.12}$$

where $k = n(\bar{k}+1)$. Using the fact that $E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] = AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right]$ and substituting equation (1.A.7), one can find:

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - K_tC\right)AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + K_tCAx_{t-1}^{(0:\bar{k})} + K_tC(B_\varepsilon\varepsilon_t + B_u u_t) + K_t[D_v v_{it} + D_u u_t] \tag{1.A.13}$$

I follow the literature by focusing in the stationary equilibrium. Therefore, the expectation of each individual i in the stationary equilibrium is the one which the MSE is in steady-state, i.e., firms update their forecast based on the steady-state Kalman gain. In other words, the dynamics of expectations depends only in the properties of the process they are forecasting and do not depend in the period t. Using equations for $P_{t+1/t}$, $P_{t/t}$ and K_t one can find the

Riccatti equation

$$P_{t+1/t} = A\left[P_{t/t-1} - P_{t/t-1}C'\left[CP_{t/t-1}C' + \Sigma_{obs}\right]^{-1}\left[CP_{t/t-1} + \Sigma_{cov}\right]\right]A' + \Sigma_{state} \quad (1.A.14)$$

Therefore, one need to iterate this equation to find the steady-state MSE, $\bar{P}$, and compute its counterpart Kalman gain, $\bar{K}$. Nimark (2017) shows that if is x_t stationary process, then the expectations hierarchy about this process, $x_t^{(0:\bar{k})}$, is also stationary. This and the fact that Σ_ε is positive definite, then there exists a steady-state solution such that $\bar{P} = P_{t+1|t} = P_{t|t-1}$ which implies the steady-state Kalman gain $\bar{K} = K_t = K_{t-1}$ (see Hamilton; 1995, chap. 13).

The individual expectation in equation of Proposition 1.3 is one in equation (3.A.8) above using the steady-state Kalman gain, $\bar{K}$. Moreover, the first order expectation is easily computed by:

$$\begin{aligned} E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] &\equiv \int_0^1 E_{it}\left[x_t^{(0:\bar{k})}\right]di \\ &= \left(I_k - \bar{K}C\right)AE_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}(CB_u + D_u)u_t \end{aligned} \quad (1.A.15)$$

□

Now I turn to verify the claim that the expectations hierarchy has dynamics guessed in equation (1.34). It is straightforward to see that $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = x_t^{(1:\bar{k}+1)}$. By definition, any order higher than $\bar{k}$ is not relevant for the equilibrium. Then, without loss of generality, I can set $E_t^{(s)}[x_t] = 0$ if $s > \bar{k}$. Therefore, one can rewrite $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$ as

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \begin{bmatrix} x_t^{(1:\bar{k})} \\ x_t^{(\bar{k}+1)} \end{bmatrix} = \begin{bmatrix} x_t^{(1:\bar{k})} \\ 0_{n\times 1} \end{bmatrix} = \begin{bmatrix} 0_{n\bar{k}\times n} & I_{n\bar{k}} \\ 0_{n\times nk} & 0_{n\times n\bar{k}} \end{bmatrix}\begin{bmatrix} x_t^{(0:k-1)} \\ x_t^{(k:\bar{k})} \end{bmatrix} = Tx_t^{(0:\bar{k})}, \quad (1.A.16)$$

where in the second equality I used the definition of $x_t^{(1:\bar{k}+1)}$ and in the third equality I used that $E_t^{(\bar{k}+1)}[x_t] = 0$, without loss of generality. In the last equality, T is defined accordingly. Therefore, the first-order expectation of expectations hierarchy is a linear transformation of the expectations hierarchy itself, given by the matrix T. By the same token, $x_t^{(0:\bar{k})}$ can be rewritten as a linear combination of x_t and $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$ such that

$$x_t^{(0:\bar{k})} \equiv \begin{bmatrix} x_t \\ x_t^{(\bar{1}:\bar{k})} \end{bmatrix} = \begin{bmatrix} I_n \\ 0_{n\bar{k}\times n} \end{bmatrix} x_t + \begin{bmatrix} 0_{n\times n\bar{k}} & 0_{n\times n} \\ I_{n\bar{k}} & 0_{n\bar{k}\times n} \end{bmatrix} \begin{bmatrix} x_t^{(1:\bar{k})} \\ x_t^{(\bar{k}:\bar{k}+1)} \end{bmatrix} = e_x' x_t + T' E_t^{(1)} \left[x_t^{(0:\bar{k})} \right], \tag{1.A.17}$$

where e_x is defined in the same way as before. Substituting the first order expectation from equation (3.A.10) into the identity above:

$$x_t^{(0:\bar{k})} \equiv e_x x_t + T' \left(I_k - \bar{K}C \right) A E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] + T'\bar{K}CAx_{t-1}^{(0:\bar{k})} + T'\bar{K}CB_\varepsilon \varepsilon_t + \bar{K}(CB_u + D_u)u_t \tag{1.A.18}$$

Then, substituting equation (1.22) and using fact that $x_t = e_x x_t^{(0:\bar{k})}$ one can find that

$$x_t^{(0:\bar{k})} \equiv e_x' \left(A_1 e_x x_{t-1}^{(0:\bar{k})} + \varepsilon_t \right) + T' \left(I_k - \bar{K}C \right) A E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] + T'\bar{K}CAx_{t-1}^{(0:\bar{k})} + T'\bar{K}CB_\varepsilon \varepsilon_t + T'\bar{K}(CB_u + D_u)u \tag{1.A.19}$$

Using the order transformation from equation (3.10) at period $t-1$ and rearranging I find that

$$x_t^{(0:\bar{k})} = \left[e_x' A_1 e_x + T' \left(I_k - \bar{K}C \right) AT + T'\bar{K}CA \right] x_{t-1}^{(0:\bar{k})} + \left[T'\bar{K}CB_\varepsilon + e_x' \right] \varepsilon_t + \left[T'\bar{K}(CB_u + D_u) \right] u_t. \tag{1.A.20}$$

Therefore, the expression above verify that $x_t^{(0:\bar{k})}$ follows the guessed form and the square brackets terms provide identities for A, B_ε and B_u such that

$$\begin{aligned} A &= e_x A_1 e_x' + T' \left(I_k - \bar{K}C \right) AT + T'\bar{K}CA \\ B_\varepsilon &= T'\bar{K}CB_\varepsilon + e_x' \\ B_u &= T'\bar{K}(CB_u + D_u) \end{aligned} \tag{1.A.21}$$

Finally, by the last equation if $D_u = 0_{q\times p_u}$, the fixed point solution for B_u is $0_{k\times p_u}$. This means that in the case of no public noise signals, the expectation hierarchy process becomes the same as in equation (3.8). Moreover, if $D_u = 0_{q\times p_u}$ is a matrix of zeros, $\Sigma_{cov} = 0_{k\times q}$. Therefore, the steady-state Kalman gain from Proposition Proposition 1.3 becomes the same as in the Proposition 1.2.

1.A.3 Model solution

The imperfect common knowledge model consists in a the Euler equation, the Taylor rule and the Phillips curve (equations 1.11, 3.68 and 1.14, respectively). They can be easily written as the following system of linear equations

$$F_1\mathbb{E}_t\left[Y_{t+1}\right]+F_2E_t^{(1)}\left[Y_{t+1}\right]+G_1Y_t+G_2E_t^{(1)}\left[Y_t\right]+L_1E_t^{(1)}\left[x_{t+1}\right]+L_2E_t^{(1)}\left[x_t\right]+M_1x_t=0, \quad (1.A.22)$$

where Y_t is a $m \times 1$ vector of endogenous variables and x_t is a $n \times 1$ vector of *unobservable* exogenous shocks. $\mathbb{E}_t[\cdot]$ is the full information expectation operator and $E_t^{(1)}[\cdot]$ is the average expectation of agents with private information.

Here for convenience I rewrite the guessed solution

$$Y_t = Qx_t^{(0:\bar{k})}. \quad (1.A.23)$$

Using the standard undermined coefficients method, one can find a solution for Q. First, computing the full information expectation of Y_{t+1} such that

$$\begin{aligned}\mathbb{E}_t\left[Y_{t+1}\right] &= Q\mathbb{E}_t\left[x_{t+1}^{(0:\bar{k})}\right] \\ &= QAx_t^{(0:\bar{k})},\end{aligned} \quad (1.A.24)$$

where in the last equality it was used the fact that $\mathbb{E}_t\left[E_t^{(k)}\left[x_t\right]\right] = E_t^{(k)}\left[x_t\right]$ for any k.[21] By computing the individual expectation and integrating on i one can find the average expectation from agents with idiosyncratic information such as

$$\begin{aligned}E_t^{(1)}\left[Y_t\right] &= QE_t^{(1)}\left[x_t^{(0:\bar{k})}\right] \\ &= QTx_t^{(0:\bar{k})},\end{aligned} \quad (1.A.25)$$

where the last equality uses the order transformation matrix that relates the expectation of hierarchy to expectations hierarchy itself. Then, one can find the average expectation for endogenous variables in $t+1$:

$$\begin{aligned}E_t^{(1)}\left[Y_{t+1}\right] &= QE_t^{(1)}\left[x_{t+1}^{(0:\bar{k})}\right] \\ &= QATx_t^{(0:\bar{k})},\end{aligned} \quad (1.A.26)$$

[21] This come from the fact that the information set of the full information agents contain the information set of private information agents. One can think in the full information agent as a one that receives all private signals, which would reveal the unobservable shocks x_t.

where the last equality uses that $E_t^{(1)}\left[x_{t+1}^{(0:\bar{k})}\right] = AE_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$. Substituting the expectations (1.A.24-3.16) in the equation (3.1) one can find:

$$\left[G_1Q + F_1QA + G_2QT + F_2QAT + (L_1A_1 + L_2)e_x'T + M_1e_x'\right] x_t^{(0:\bar{k})} = 0_{m\times n} \tag{1.A.27}$$

Since this equality is valid for any value of $x_t^{(0:\bar{k})}$, the only way it is always satisfied is if the term in square brackets is equal to zero. The solution for Q can be found by straightforward vectorization. Note, however, that unlike the full information model and the standard imperfect information model, the solution Q depends on A which in turn is a function of Q through the matrix C of the observation equation. Since firms observe an endogenous variable, the observational equation depends on the response of endogenous variables to the expectations hierarchy (recall equations 1.26 and 1.35). Therefore, the model solution is a fixed point of the solution to equations (1.A.27) and (1.A.21).

Following a similar approach of Nimark (2008) and Melosi (2017), the fixed-point solution can be found by the algorithm:

Set the initial values $(A^{(0)}, B_\varepsilon^{(0)}, B_u^{(0)})$ and a small tolerance $\epsilon > 0$ and set $i = 1$. Then, follow the steps:

1. given $A = A^{(i-1)}$, solve for Q the equation (1.A.27) by vectorization. Set $Q^{(i)} = Q$.
2. Given $Q^{(i)}$, construct the matrix C from the observation equation (1.A.6). Set $C^{(i)} = C$.
3. Given $B_\varepsilon^{(i-1)}$, $B_u^{(i-1)}$ and $C^{(i)}$ compute the Kalman gain, K_t and $P_{t+1|t}$ using equations (3.A.4) and (3.A.5), respectively. Set $K^{(i)} = K_t$ and $P^{(i)} = P_{t+1|t}$.
4. Given $B_\varepsilon^{(i-1)}$, $B_u^{(i-1)}$, $A^{(i-1)}$, $C^{(i)}$ and $K^{(i)}$, compute the B_ε, B_u, A using equation (1.A.21). Set $B_\varepsilon^{(i)} = B_\varepsilon$, $B_u^{(i)} = B_u$, $A^{(i)} = A$.
5. if $\min \|B_\varepsilon^{(i)} - B_\varepsilon^{(i-1)}\| < \epsilon$, $\|B_u^{(i)} - B_u^{(i-1)}\| < \epsilon$, $\|A^{(i)} - A^{(i-1)}\| < \epsilon$ and $\|P^{(i)} - P^{(i-1)}\| < \epsilon$, stop iterating or else set $i = i + 1$ and go back to step 1.

1.A.4 Decomposition of Impulse Response Functions

For convenience, I rewrite the solution of the ICI model given by

$$Y_t = Q_1 x_t + Q_2 E_t[x_t]. \tag{1.A.28}$$

by the definition of x_t, one can rearrange this equation as

$$Y_t = Q_{1,\eta}\eta_t + Q_{1,\pi^*}\pi_t^* + Q_{2,\eta}E_t[\eta_t] + Q_{2,\pi^*}E_t[\pi_t^*], \tag{1.A.29}$$

where Qi, j are conformable submatrices of Q_i such that $i = \{1, 2\}$ and $j = \{\pi^*, \eta\}$. Therefore, this four terms perfectly decompose the response of endogenous variables to shocks. For instance, in the decomposition of the impulse response function a standard deviation monetary shock shown in Figure 1.4, I set $\varepsilon_0^\eta = \sigma_\eta = 0.15$, $\varepsilon_t^\eta = 0$ for $t \geq 1$ and $\varepsilon_t^{\pi^*} = 0$ for all t. Then, shocks and their expectations evolve accordingly to equations (1.22) and (1.23), respectively. Then, first term standards for the "Actual shock", the second equals zero, the third for "Monetary shock expectations" and the last one for "Inflation shock expectations". For the inflation target shock it works analogous.

For the ICK model, for convenience, I write the model solution given by

$$Y_t = Qx_t^{(0:\bar{k})} \tag{1.A.30}$$

by the definition of x_t and $x_t^{(0:\bar{k})}$ one can rearrange this equation as

$$Y_t = Q_{0,\eta}\eta_t + Q_{0,\pi^*}\pi_t^* + \sum_{k=1}^{\bar{k}} Q_{k,\eta}E_t^{(k)}[\eta_t] + \sum_{k=1}^{\bar{k}} Q_{k,\pi^*}E_t^{(k)}[\pi_t^*] \tag{1.A.31}$$

where Qk, j are conformable submatrices of Q such that $k = 1, 2, ..., \bar{k}$ and $j = \{\pi^*, \eta\}$. Again, this four terms decompose the response to shocks. Then, decomposition of Figure 1.4, the innovations are defined as describe above and shocks and their higher-order expectations evolve accordingly to equations (1.22) and (3.8), respectively. Then, first term standards for the "Actual shock", the second equals zero, the third for "Monetary shock hierarchy" and the last one for "Inflation shock hierarchy". For the inflation target shock it works analogous. I chose to decompose the IRFs in only three terms: the impact of the actual shock, of all higher-order expectations to the monetary shock and all higher-order expectations to the target shock. One could alternatively decompose it also by each order of each shock.

Chapter 2

On the implications of strategic uncertainty for expectation formation in macroeconomics

Abstract

This paper study the implications of the strategic uncertainty for imperfect information models. I compare the learning process under two assumptions. Under imperfect information, agents solve a standard signal extraction problem given their observables. Under imperfect common knowledge, agents observe the same variables but also noisy private signals about the shocks.

I show that whenever agents extract information from endogenous variables that depend on the underlying unobserved shock, the signal extractions from those variables under imperfect information and imperfect common knowledge are different. This difference in learning about unobserved shocks does not vanish even in the limiting case when the variance of the private signal goes to infinity. Intuitively, strategic uncertainty prevents agents from knowing other agents' decision, despite that those actions are the same in equilibrium. Therefore, agents are unable to infer aggregate outcomes from their own choices, which affects their ability to extract information from those endogenous variables.

This discontinuity result challenges this benchmark assumption by showing the substantial knowledge about endogenous variables implicitly assumed available to agents in imperfect information models. The results imply that one should assume that agents cannot infer aggregate outcomes from individual decisions whenever individuals acquire information from endogenous variables.

Keywords: Signal extraction, imperfect information, imperfect common knowledge.
JEL classification: E32, D82, D83, D84

2.1 Introduction

There is a long tradition of imperfect information models in Macroeconomics that goes back to the rational expectations revolution in the 1970s. Lucas (1972) proposed a rational expectations model which imperfect information about economic conditions are key to give rise a short-term relationship between inflation and real output. In that model, agents face uncertainty whether their price was changing because of relative price or aggregate price changes. Despite this initial interest, due to the inability of those models to explain the dynamics of the main macroeconomic aggregates, macroeconomic literature focused into full information models with other frictions such as price and wage rigidity.

A more recent literature starting with Mankiw and Reis (2002), Woodford (2002) and Sims (2003) renewed the interest in better understanding information frictions. In comparison to the seminal literature, there are three main differences (see Mankiw and Reis (2010) for a discussion).

First, instead of the information being perfectly revealed after an initial period, agents learn gradually about the economy. Gradual learning can be a result of agents infrequently updating their information due to limitations processing or acquiring information as well due to perfectly updated but noisy information. Persistence in expectation formation induces inertia into the endogenous variables, which can help to explain the dynamics of the key macroeconomic aggregates.

Second, there is a more significant emphasis on information dispersion. A notable exception is the seminal work of Townsend (1983) that shows that when agents extract a signal about an underlying fundamental from aggregate outcomes, there is an infinite-regress of forecasting the forecast of others.

Third, there is emphasis the strategic interaction of agents with imperfect information. The recent literature starting with Morris and Shin (2002), strategic uncertainty plays an essential role in the analysis. Unlike the earlier contribution of Townsend (1983), higher-order expectations matter because of strategic complementarities (and not because agents learn from an aggregate endogenous outcome).

This paper emphasizes the implications of strategic uncertainty for imperfect information models by comparing the expectation formation under two important classes of models of the recent literature. Under the first assumption, agents observe a set of endogenous public signals about unobservable shocks. Therefore, those agents solve the same signal extraction problem. Under the second assumption, agents observe the same set of endogenous variables but also receive exogenous idiosyncratic noisy signals about the shocks. This implies that agents do not know the shocks hitting the economy, but they also do not know the assessment of other

agents about those shocks. I follow the literature referring this assumption by imperfect common knowledge (see, for instance, Woodford; 2002; Adam; 2007; Nimark; 2008).

Previous literature compared the properties of those models with the full information case as a benchmark but not with each other. This happens because of the well-known result that the imperfect information is a particular case of the imperfect common knowledge when the variance of the idiosyncratic signal is unbounded large. In the following, I will call this case as a non-informative idiosyncratic signal.

The main result is as follows. I show that the imperfect information model is a particular case of the imperfect common knowledge model if and only if the signal extraction is from exogenous signals. When agents learn from endogenous variables, I show the conditions that break down this well-known result. Whenever agents extract information from endogenous variables that depends on the underlying unobserved shock (either directly or via a relationship with other endogenous variable), the expectation formation of both models are different even in the limiting case of an unbounded variance for the idiosyncratic signal. However, if agents observe an endogenous variable that depends only on expectations of the unobservable shock, then the equilibrium in the limiting case converges to the model under imperfect information.

The intuition is as follows. Under dispersed information, since agents do not share the same information, they do not know other agents' decisions in equilibrium, i.e., there is strategic uncertainty. This prevents agents from knowing other agents' decisions, which implies that those agents cannot infer aggregate variables from their own decisions. The same argument holds in the limiting case of a non-informative idiosyncratic signal. In that case, all agents have the same information about the unobserved shock (and have the same optimal decisions) but still do not know other agents' decisions, i.e., strategic uncertainty still holds.

If those agents extract information from an endogenous variable that depends on the aggregate outcome, this changes the ability to extract information from that observable. Therefore, the signal extraction is different than under imperfect information and common knowledge. Since all agents share the same information, extracting information from an endogenous variable that responds only to expectations does not provide any additional information. There is additional information only if that variable incorporates heterogeneous information of other agents, which does not the happen in the limiting case.

In other words, under imperfect information, agents are all equal, and they know that. Under imperfect common knowledge in the limiting case of a non-informative private signal, agents are equal, but they do not know that. If those agents observe a variable that aggregates their behavior to extract information about an underlying fundamental, the fact that they

know if they are the same or not matters.

The difference in learning is not trivial. It can qualitatively change the responses to structural shocks as well to the noises of the public signals. For concreteness, I first apply the main results to a simple island model where firms do not observe the money supply and extract information from public and private signals. This public signal is noisy and reacts to prices, output, and the money supply (e.g., interest rates or asset prices). The model results in a simple Lucas' supply curve such that expected money supply changes affect only prices whereas unexpected changes affect both prices and output.

In the imperfect information case, the dynamics of prices and output as well their relationship to expected and unexpected changes in money supply does not depend on the reaction of the public signal to output and prices (since output and prices are known under that assumption). Under imperfect common knowledge with a non-informative private signal, the responsiveness of the public signal to output affects the price and output outcomes. The reaction of the public signal to prices does not play a role.

The intuition is the following. Output responds directly to the money supply. If the public signal responds more to output, for a given variance of the noise of the signal, it is also more informative about the money supply. The price level responds only to expectations about the money supply. In the limiting case, all agents have the same information. Observing the price level only provide information that agents already have. Thus, the being more responsive to prices does not affect the informativeness of the public signal.

In a particularly extreme case, depending on how the public signal reacts to output, it can completely break down the relationship between money, output, and prices. However, in general, the difference in learning process does not affect this relationship qualitatively. The reaction to output by the public signal can change qualitatively the responses to the its noise. More recent literature has emphasized this noise shocks as a potential driver of business cycles (see, for instance, Lorenzoni; 2009; Angeletos and La'O; 2010; Barsky and Sims; 2012; Blanchard et al.; 2013).

This simple model highlights that imperfect information models can be misleading if agents are extracting information from publicly available macroeconomic data.

2.2 Related literature

The first class of models considered – imperfect but common information – rely in a dynamic signal extraction problems following the seminal contribution of Lucas (1972). For instance, in Boz et al. (2011), Barsky and Sims (2012) and Blanchard et al. (2013) agents

cannot distinguish permanent and temporary shocks in productivity. The former discusses that higher uncertainty regarding the growth trend in emerging markets can explain the relatively higher consumption variability than in developed countries. The others assess the relative importance of news and noise to business cycle fluctuations empirically.

This learning process is also used to understand the implications of shifts in the conduct of monetary policy. Erceg and Levin (2003) introduces a model in which agents cannot disentangle persistent shifts in the inflation target from transitory monetary shocks. They show that the learning process can account for realistic disinflation costs.

Andolfatto et al. (2008) discuss the implications of this learning process for the inflation expectations formation. They show that the model with rational agents with imperfect information can generate biased expectations and persistent forecast errors which are consistent with expectation survey data. Del Negro and Eusepi (2011) compare the empirical fit of the imperfect information model relative to the full information case. Despite incorporating those features, they find that the imperfect information model is not superior in explaining the inflation expectations data than the full information benchmark.

Models with imperfect common knowledge model structure also have been used to explain a variety of phenomena. Woodford (2002) shows that imperfect common knowledge about the nominal GDP is an alternative to sticky prices to understand the real effects of monetary shocks. Strategic uncertainty can induce significant inertia in response to monetary shocks through the slow adjustment of higher-order expectations. Adam (2007) shows the optimal monetary policy when agents have imperfect and dispersed information about supply and demand shocks in a static model. Lorenzoni (2010) uses an RBC model that when agents have dispersed information about productivity. He shows when agents extract information from the interest rate, the optimal monetary policy is an inertial rule.

Lorenzoni (2009) and Angeletos and La'O (2010, 2013) use RBC model with imperfect and dispersed information setup to explain business cycles. In the latter, sentiment shocks are an important driver of the fluctuations whereas in the others, the key determinant of cycles is the noise on public signal about productivity.

Unlike previous papers, Nimark (2008) proposes an approximate solution method that the simplifying assumption that the unobserved underlying fundamental is observed after a period is not needed. Nimark (2017) shows that this approximating method that truncates the higher-order expectations can be as accurate as one wants for a class of models that the importance of higher-order expectations decreases with the order.

This paper relies on an extension of this solution method proposed in Chapter 1 of this thesis, that allows not only endogenous variables and private signal into the agents'

information sets but also for public signals. Moreover, was the first to point out of a qualitative different learning process for imperfect and dispersed information models.

In Chapter 1, I use a New Keynesian model where agents extract information about the monetary and inflation target shocks by observing the interest rate. In that case, strategic uncertainty prevents firms from observing the inflation and output, which are key determinants of the interest rate. It turns out that this leads to a qualitative different learning process about the inflation target. Whether agents observe of not those variables affect the ability to extract information from the interest rate. This paper generalizes this result for a general class of DSGE models with unobservable shocks that extract information from an endogenous variable.

In section 2.3, I show the implications of strategic uncertainty to a well known Lucas-Phelps type of model. Section 2.4 show that the results apply to a general class of DSGE models.

2.3 Simple model

There is a representative household that choose consumption and labor supply to maximize her utility function. There also is a continuum $i \in [0,1]$ of monopolistic competition firms that do not observe the money supply m_t. Prices and wages are fully flexible. Firms maximize profits choosing prices subject to a (log-linear) isoelastic demand from households.

Given the equilibrium in good and labor markets, the optimal price choice of firms is given by

$$p_{it} = \alpha E_{it}[p_t] + (1-\alpha)E_{it}[m_t], \tag{2.1}$$

where $\alpha \in (0,1)$ is a parameter that measures strategic complementarities in price decisions. $p_t = \int_0^1 p_{it}di$ denote the log-linear price index and m_t is the money supply. $E_{it}[\cdot] \equiv E[\cdot|\mathcal{I}_{it}]$ is the firm i conditional expectation operator and $\mathcal{I}_{it}$ is firm i information set at period t.

Since the optimal solution in the form of equation (2.1) is well-known, I refer the reader to the Appendix 2.A.1 for the microfoundation of this equation.

The demand side of the economy is a exogenous stochastic process for the nominal demand is given by

$$q_t \equiv y_t + p_t = m_t + v_t, \tag{2.2}$$

where $v_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is money velocity shock and $y_t = \int_0^1 y_{it}di$ denote the (real) aggregate

demand. Moreover, suppose that the money supply is also stochastic process such that

$$m_t = \rho m_{t-1} + \varepsilon_t, \tag{2.3}$$

where $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$. Given this simple demand side, anything is assumed about the monetary transmission mechanism.

The information setup is as follows. The money supply is not direct observed by firms. They differ only by their information about the money supply. Note that money velocity shock prevents firms from observing m_{t-1} at period t even if firms are able to observe last period output, y_{t-1}, and prices, p_{t-1}. Therefore, the whole history of m_t is unknown.

I assume that each firm i observe a private signal given by

$$x_{it} = m_t + w_{it}, \tag{2.4}$$

where $w_{it} \sim \mathcal{N}(0, \sigma_w^2)$. Additionally, each firm i observes a public signal such that

$$z_t = \phi_p p_t + \phi_y y_t + m_t + u_t, \tag{2.5}$$

where $u_t \sim \mathcal{N}(0, \sigma_u^2)$. The parameters ϕ_p and ϕ_y determine how the public information depends on both price index and aggregate demand. One can think z_t as a indicator widely available that depends on the money supply and the state of the economy, for instance, interest rates.

In comparison with the standard imperfect information model there are two additional ingredients: the private signal, x_{it}, and the endogenous public information, i.e., $\phi_p \neq 0$ and $\phi_y \neq 0$. If we ignore the public information, the model presented is essentially same as the one from Woodford (2002).[1] If we consider a exogenous public signal (i.e., $\phi_p = \phi_y = 0$) and a static fundamental ($\rho = 0$), the model is similar to the one in Morris and Shin (2002).

In the following, I first solve the standard imperfect information and then compare with the imperfect common knowledge model.

2.3.1 Imperfect information

Under imperfect information, every firm i observe only the public signal z_t. Since information is common and there is no other type of heterogeneity, all firms choose the same price, i.e., $p_{it} = p_t$ for $\forall i \in [0, 1]$. Therefore, the optimal choice from equation (2.1) becomes

[1] Woodford (2002) uses a exogenous AR(1) process in the growth rate of Nominal GDP. For simplicity, I use a AR(1) in the money supply with a money velocity shock.

simply

$$p_t = E_t(m_t). \tag{2.6}$$

Moreover, by substituting this equation into the aggregate demand (2.2), it is easy to see that

$$y_t = m_t - E_t[m_t] + v_t. \tag{2.7}$$

Therefore, in order to pin-down the equilibrium price and real output, one have to find the expectation and unexpected changes in money supply, respectively.

In the imperfect information model, agents observe the whole history of endogenous variables. The price and output provide information the money supply through their role in the public signal. More specifically, firms compute the *adjusted public signal*, $\hat{z}_t$, given by

$$\hat{z}_t \equiv z_t - \phi_p p_t - \phi_y y_t = m_t + u_t. \tag{2.8}$$

Therefore, the signal extraction of this model can be computed by Kalman filter using equation (2.3) as state equation and equation (2.8) as observational equation. The following proposition shows how firms form expectations about the money supply.

Proposition 2.1. *The rational expectations about money supply and unanticipated change in money supply are given by*

$$E_t[m_t] = (1 - \bar{k})\rho E_{t-1}[m_{t-1}] + \bar{k}\rho m_{t-1} + \bar{k}(\varepsilon_t + u_t), \tag{2.9}$$

$$m_t - E_t[m_t] = (1 - \bar{k})\rho(m_{t-1} - E_{t-1}[m_{t-1}]) + (1 - \bar{k})\varepsilon_t - \bar{k}u_t, \tag{2.10}$$

respectively. $\bar{k} \in [0, 1]$ *is the steady-state Kalman gain that solves the system:*

$$\begin{aligned} \bar{k} &= \frac{\bar{p}}{\bar{p} + \sigma_u^2} \\ \bar{p} &= \rho^2(1 - \bar{k})\bar{p} + \sigma_\varepsilon^2, \end{aligned} \tag{2.11}$$

where $\bar{p}$ *is the steady-state mean squared error of the one-period ahead forecast of* m_t*.*

The proof is a simple application of the Kalman filter and is left in the Appendix 2.A.2. This proposition states that evolution of the expectations as function of the previous expectation and the realization of m_t and u_t.

Note that the dynamic signal extraction problem induces persistence in the expectation formation as shown by equation (2.9). In turn, this implies that the unanticipated change in money supply is also persistent.

The well-known result from Lucas (1972) applies: unanticipated changes in money supply both affect aggregate output and prices, whereas expected changes are all incorporated in prices. There is a important qualification, though. The gradual learning process about the money supply implies that past unanticipated changes have effects on aggregate output and prices and past expected changes are incorporated in current prices.

To see this, by iterating backwards equation (2.9) and substituting into the optimal price (2.6) one can see that

$$p_t = \bar{k} \sum_{l=0}^{\infty} (1 - \bar{k})^l \rho^l (m_{t-l} + u_{t-l}). \tag{2.12}$$

Likewise, by iterating backwards the forecast error in equation (2.10) substituting this equation into the aggregate demand (2.2), the equilibrium y_t is given by

$$y_t = \sum_{l=0}^{\infty} (1 - \bar{k})^l \rho^l \left((1 - \bar{k}) \varepsilon_{t-l} - \bar{k} u_{t-l} \right) + v_t. \tag{2.13}$$

The current unexpected changes result from both ε_t and u_t, i.e., variation in money supply or pure noise from the public signal that affects endogenous variables only through expectations. But also the past unexpected changes (ε_{t-l} and u_{t-l} with $l > 0$) also matters. The more persistent the money supply process is (higher ρ) or less informative the public signal is (lower $\bar{k}$), the higher the effect of past expectation error on current output and prices.

For any t, an unexpected increase in money supply ($\varepsilon_t > 0$) increases both the price level and output. Note that, for any t, an increase in u_t affects output and prices in different directions. The intuition is the following. For any t, an higher u_t implies that agents expect will expect a rise in money supply in period t, since they observe an higher adjusted public signal, z_t. This implies an higher price level owing to expectations of a growing money supply. However, since the money supply which did not actually change, there is an negative surprise in money supply, which decreases output.

2.3.2 Imperfect common knowledge with non-informative private signal

In the imperfect common knowledge model, firms observe both signals, private and public. The idiosyncratic signal introduces strategic uncertainty on pricing decisions. In other words, when choosing prices, firms do not know other firms' price decisions. I also assume the following timing. Each period t, at the beginning of the period, firms receive their signals (x_{it}, z_t) and choose prices given their information. Then, in the end of the period, firms production accommodate the households demand and labor supply. This simplifies the analysis by preventing the introduction of other endogenous source of information such as wages or individual production. Each firm i optimal pricing equation is given by (2.1). They have to form expectations about the aggregate price and money supply. The expectation $E_{it}(\cdot) \equiv E(\cdot|\mathcal{I}_t^i)$ is the firm i expectation given its information set, $\mathcal{I}_t^i = \{x_{is}, z_s | s \leq t\}$. Moreover, the price index is computed by aggregating individual price decisions, i.e., by integrating equation (2.1) such that

$$p_t = \alpha E_t^{(1)}[p_t] + (1-\alpha) E_t^{(1)}[m_t], \tag{2.14}$$

where $E_t^{(1)}[\cdot] = \int_0^1 E_{it}[\cdot]di$ is the first order average expectation. It will be useful to define the k-th order expectation $E_t^{(k)}[\cdot]$ given information of period t about a variable x_t

$$E_t^{(k)}[x_t] = \int_0^1 E_{it}\left[E_t^{(k-1)}[x_t]\right] di, \tag{2.15}$$

for $k \geq 1$, with the convention $E_t^{(0)}[x_t] = x_t$. Computing the expectation of equation (2.14) and substituting back iteratively, one can find that the price index can be rewritten as

$$p_t = (1-\alpha) \sum_{k=1}^{\infty} \alpha^{k-1} E_t^{(k)}[m_t]. \tag{2.16}$$

This equation shows that the aggregate price is a weighted average of the higher-order average expectations about the money supply. If agents had the same information, higher-order expectations would collapse to the common expectation, and equation (2.16) would become the same as price level of the imperfect information model (equation 2.6).

The strategic uncertainty prevents firms from knowing current price index, p_t, and the aggregate output, y_t, since firms do not know other firms' price and production decisions. This is key for firms extracting information about m_t from the public signal z_t. Consequently, firms have to consider not only the direct effect of m_t on z_t (as in the imperfect information case), but the indirect effect of m_t and its higher-order expectations through p_t and y_t. Formally,

substituting the aggregate demand (2.2) into the the public signal (2.5), one can find

$$z_t = (1 + \phi_y)m_t + (\phi_p - \phi_y)p_t + \phi_y v_t + u_t. \tag{2.17}$$

Moreover, substituting price level (2.16) into the public signal

$$z_t = (1 + \phi_y)m_t + (\phi_p - \phi_y)(1 - \alpha)\sum_{k=1}^{\infty} \alpha^{k-1} E_t^{(k)}[m_t] + \xi_t \tag{2.18}$$

where $\xi_t = u_t + \phi_y v_t$. This equation shows how the public signal depends on the money supply and its higher-order expectations.

Woodford (2002) shows how to find an exact solution to this problem but with exogenous information ($\phi_p = \phi_y = 0$). Instead of pursuing an extension for this exact solution, I use the approximate solution method of Chapter 1 that allows endogenous variables, public and private signals as observables. I use this method because the exact solution method is not available for the general case. Applying this method is without loss of generality since in the limiting case of $\sigma_w \to \infty$, the solution is exact.[2]

Consider the limiting case that $\sigma_w \to \infty$, which I call the case of a *non-informative private signal*. In that case, since firms will not update information based on x_{it}, it is equivalent, for the matter of computing the equilibrium, the two following approaches. i) finding the expectations dynamics through the Kalman filter and applying the limit $\sigma_w \to \infty$ to find the equilibrium Kalman gain; ii) using the Kalman filter ignoring the private signal x_{it}, i.e., consider only the public signal, z_t, as observable.

For this simple case, I follow the second approach that is easier to compute, but I provide a formal proof for this claim in the general model that encompasses this simple case. The intuition for this result is that, in this limiting case, the signal is sufficiently noisy such that it not provide any useful information about m_t, and thus, the same equilibrium can be computed by not considering it.

In this limiting case, the only source of information is the public signal, in the same manner as in the imperfect information model. Therefore, one would think that both models must provide the same expectations and equilibrium outcomes. In the following, I show that it is not always the case that this is true.

Since the private signal x_{it} is the only source of heterogeneity in expectations, by ignoring it, all firms must have the same expectation, i.e., $E_{it}[m_t] = E_{jt}[m_t] = \hat{E}_t[m_t]$ for any $i, j \in [0, 1]$. This implies that the law of iterated expectations holds as agents share the

[2]As it is going to be clear later, the approximate solution method rely on a truncation of the number of higher-order expectations considered. In the limiting case, all higher-order expectations collapse to a common expectation. Therefore, the solution considering only the first-order expectation is exact.

same information (see Morris and Shin (2005) for a discussion). Therefore, the higher-order expectations are also equal to the common expectation, i.e., $E_t^{(k)}[m_t] = \hat{E}_t[m_t]$ for all k. In that case, the pricing equation simplifies to[3]

$$p_t = \hat{E}_t[m_t], \tag{2.19}$$

where, formally, $\hat{E}_t[\cdot] \equiv \lim_{\sigma_w \to \infty} E[\cdot|x_{it}, z_t]$.

If the expectations formation in this limiting case is the same of the one in the imperfect information model, the equilibrium of both would be the same. By the same reason, the equation (2.18) can be also simplified to

$$z_t = (1 + \phi_y)m_t + (\phi_p - \phi_y)\hat{E}_t[m_t] + \xi_t. \tag{2.20}$$

The comparison of the public signal in equation (2.20) with the *adjusted public signal* from equation (2.8) highlights the differences in information content from both signals. The latter has a one-to-one relation to the money supply and has a noise with variance equals to σ_u^2. The former have a different coefficient on m_t, a noise with higher variance, $\sigma_u^2 + \phi_y^2\sigma_v^2$, and also depends on the expectation of m_t. Despite that all firms in equilibrium have the same information and the same choice, firms are not aware of that. Within this limit, strategic uncertainty is still present, and thus, firms do not know the price level, p_t, and the aggregate output, y_t.

In the following, I guess (and verify) that the expectation about the money supply, $\hat{E}[m_t]$, follows

$$\hat{E}_t[m_t] = \rho_1 m_{t-1} + \rho_2 \hat{E}_{t-1}[m_{t-1}] + b_\varepsilon \varepsilon_t + b_\xi \xi_t \tag{2.21}$$

where $\rho_1, \rho_2, b_\varepsilon$ and b_ξ are coefficients to be determined in equilibrium. The following proposition shows how the rational expectation is computed in this case.

Proposition 2.2. *Consider the limiting case of a non-informative private signal ($\sigma_w \to \infty$). In that case, the rational expectations about money supply and unanticipated change in money supply are given by*

$$\hat{E}_t[m_t] = \rho_1 m_{t-1} + \rho_2 \hat{E}_{t-1}[m_{t-1}] + b_\varepsilon \varepsilon_t + b_\xi \xi_t, \tag{2.22}$$

[3]Note that agents still do not observe p_t, which is the not case under imperfect information. However, since the expectations are common among agents, the law of iterated expectations holds. Therefore, by taking expectations in equation (2.1) and rearranging, one can find that $\hat{E}[p_t] = \hat{E}[m_t]$. By substituting it back, one can find the equation (2.19).

$$m_t - \hat{E}_t[m_t] = \rho_2 \left(m_{t-1} - \hat{E}_{t-1}[m_{t-1}] \right) + (1 - b_\varepsilon)\varepsilon_t - b_\xi \xi_t, \tag{2.23}$$

respectively. The equilibrium coefficients $(\rho_1, \rho_2, b_\varepsilon, b_\xi)$ *are such that*

$$\begin{aligned} \rho_1 &= \hat{k}_e(1 + \phi_y)\rho; \\ \rho_2 &= [1 - \hat{k}_e(1 + \phi_y)]\rho; \\ b_\varepsilon &= \hat{k}_e(1 + \phi_y); \\ b_\xi &= \hat{k}_e, \end{aligned} \tag{2.24}$$

where the effective Kalman gain, $\hat{k}_e$*, that solves the system:*

$$\hat{k} = \frac{(1 + \phi_y)\hat{p}}{(1 + \phi_y)(1 + \phi_p)\hat{p} + \sigma_\xi^2} \tag{2.25a}$$

$$\hat{k}_e = \frac{\hat{k}}{1 - (\phi_p - \phi_y)\hat{k}} = \frac{(1 + \phi_y)\hat{p}}{(1 + \phi_y)^2\hat{p} + \sigma_\xi^2} \tag{2.25b}$$

$$\hat{p} = \rho^2(1 - \hat{k}_e(1 + \phi_y))\hat{p} + \sigma_\varepsilon^2 \tag{2.25c}$$

where $\hat{p}$ *is the steady-state mean squared error of the one-period ahead forecast of* m_t*,* $\hat{k}$ *is the Kalman gain and,* $\sigma_\xi^2 = \sigma_u^2 + \phi_y^2\sigma_v^2$*.*

The proof is in the Appendix 2.A.3. The proposition above shows how firms forms expectations based on the public signal, z_t, and the private signal, x_{it}, in the limiting case that $\sigma_w \to \infty$. Several comments are noteworthy. First, the expectation dynamics is a function of the *effective Kalman gain* , $\hat{k}_e$ and not on the standard Kalman gain, $\hat{k}$.

Second, the effective Kalman gain discounts, from the standard Kalman gain, the variation from the public signal that comes from the expectations about money supply. Therefore, equation (2.25b) denominator has the discounting term $(\phi_p - \phi_y)\hat{k}$. Note that $\hat{k}$ determines how much the expectation $\hat{E}[m_t]$ responds to variations in z_t. The term $(\phi_p - \phi_y)\hat{k}$ is how much $\hat{E}[m_t]$ responds to variations of z_t that comes from itself (see equation 2.20).

The intuition is the following. Since the expectation, $\hat{E}[\cdot]$, is formed based on information that is common to all firms, variations of the public signal that come from the expectation itself only reflects information that firms already have. Thus, there is no reason to update expectations based on this particular variation of the public signal.

An alternative manner to see this, notice that the effective Kalman gain is equivalent to the Kalman gain that would arise if firms observe the following signal:

$$\tilde{z}_t \equiv z_t - (\phi_p - \phi_y)\hat{E}_t[m_t] = (1 + \phi_y)m_t + \xi_t. \tag{2.26}$$

Therefore, in this limiting case, firms learn from z_t as if they were learning from the signal $\tilde{z}_t$, that disregards the term related to the price level (which is equal to the money supply expectation). I call this *effective public signal* (which should not be confused with the adjusted public signal in the imperfect information case). Owing to this, while the Kalman gain depends on both ϕ_p and ϕ_y, the effective Kalman gain depends only on ϕ_y. Intuitively, the real output provide information about the money supply whereas the price level only provide information about the expectations of the money supply. The former provide new additional information about the money supply to firms whereas the latter only reflect information already incorporated in firms' expectations.

Moreover, and most important, the effective Kalman gain, $\hat{k}_e$, is not the same to the Kalman gain of the imperfect information model, $\bar{k}$ (equation (2.11). This implies that expectations formation under imperfect information model and under imperfect common knowledge in the limit that $\sigma_w \to \infty$ are different. Therefore, the equilibrium real output and prices are different in those models. This seems puzzling since the private signal with noise w_{it} is the only feature that differentiate both models.

In order to understand this, note first that the introduction of the private signal, x_{it}, induces strategic uncertainty. The fact that the private signal becomes non-informative does not affect the fact that firms still do not known other firms' decisions when they are forming expectations, although they choose the same in equilibrium. Since firms do not known other firms' decisions, they are not able to infer p_t and y_t. It turns out that, for the reasons describe above, the information from the prices through the public signal do not matter but the one from y_t does. By comparing the *effective public signal* from the ICK model (equation 2.26)to the *adjusted public signal* from the ICI model (equation 2.8). Those signals respond differently to m_t and are subject to different noises. Therefore, they have a different information content about the money supply.

Although most of the literature using similar models focus on implications of higher-order expectations (Morris and Shin; 2002; Woodford; 2002; Adam; 2007; Angeletos and La'O; 2009), they do not play an important role for the result here since α do not even affect the expectations and equilibrium outcomes.

Despite the differences in learning processes under imperfect information and the limiting case under imperfect common knowledge, the expectation formations are not disconnected. If $\phi_y = 0$, the limiting equilibrium from Proposition 2.2 it exactly the same from the one in Proposition 2.1. In that case, $\hat{k}_e = \bar{k}$, $\hat{p} = \bar{p}$ and $\xi_t = u_t$. In other words, as long as the

public signal responds to output, which is not observable because the strategic uncertainty, the learning process are different.

Despite the learning differences, the model in the limiting case still have the key feature from the model under imperfect information: expected money supply changes affects only prices whereas unexpected changes affect both prices and output.

However, there is one exception: the case that $\phi_y = -1$. In that case, the effective public signal is uncorrelated with the money supply. Therefore, that signal provide no information about m_t ($\hat{k} = \hat{k}_e = b_\varepsilon = b_\xi = 0$). Thus, the conditional expectation equals to the unconditional one, $\hat{E}_t[m_t] = 0$ with MSE equal to the unconditional variance, $\hat{p} = \frac{\sigma_\varepsilon^2}{1-\rho^2}$. This implies that $p_t = 0$ and $y_t = m_t + v_t$. In other words, in that extreme case, price level is unrelated to changes in money supply whereas output responds one-to-one to those changes. This extreme case highlights how different the equilibrium under imperfect common knowledge in the limiting case and under imperfect information.

For the other cases, the parameter ϕ_y is key to determine the information content of the effective public signal and thus, the dynamics of endogenous variables. Note that $(1+\phi_y)\hat{k}_e = \frac{(1+\phi_y)^2\hat{p}}{(1+\phi_y)^2\hat{p}+\sigma_\xi^2} \in (0,1]$ for all $\phi_y \neq -1$. Therefore, as long as $\phi_y \neq -1$, the sign of ϕ_y does not affect the positive relationship between money, output and prices.

This is not true for the response to the noise of public signal. ϕ_y controls how reactive is the effective public signal to the money supply, and thus, how informative it is about the money supply. This in turn, drives the reaction of the endogenous variables to ξ_t.

If $\phi_y = 0$, the expectations are the same of under imperfect information as well the responses of the endogenous variables. For $\phi_y > -1$, the public signal has positive correlation with the money supply. In that case, an increase in the public signal indicates a rise in money supply ($\hat{k}_e > 0$). This implies a increase in the price level and a output drop in response to a positive noise $u_t > 0$, i.e., a response qualitatively similar to the one in the imperfect information model. For $\phi_y < -1$, the opposite is true. An increase in the public signal indicates an drop in money supply ($\hat{k}_e < 0$), which translate drop in price level and increase in output.

Another interesting difference is that the money velocity shock, v_t, now affects the price level, p_t. Under imperfect information, it was not the case since it does not affect any information from the money supply. Under imperfect common knowledge, it affects the information content of the public signal via its variation on output (recall that $\xi_t = u_t + \phi_y v_t$).

In this section, I presented a simple model to highlight the potential differences in the learning process under imperfect information and under imperfect common knowledge. Those differences does not vanish even in the case that the private signal is non-informative. In the

next section, I show that this result do not depend on the specific details and applies to a general class of linear rational expectation with imperfect information.

2.4 General linear rational expectation model

In this section, I provide a general treatment of the results obtained in the simple example. Consider the system of rational linear equations given by

$$FE_t^{(1)}\left[Y_{t+1}\right] + G_1 Y_t + G_2 E_t^{(1)}\left[Y_t\right] + L_1 E_t^{(1)}\left[x_{t+1}\right] + L_2 E_t^{(1)}\left[x_t\right] + Mx_t = 0, \tag{2.27}$$

where Y_t is a $m \times 1$ vector of endogenous variables and x_t is a $n \times 1$ vector of unobservable exogenous shocks. $E_t^{(1)}[\cdot] \equiv \int_0^1 E_{it}[\cdot]\, di$ is the average expectation of agents $i \in [0,1]$.

Moreover, the dynamics of the shocks is given by:

$$x_t = A_1 x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \Sigma_\varepsilon) \tag{2.28}$$

where Σ_ε is a $n \times n$ diagonal matrix. Throughout this paper, I assume that the matrix A_1 has all its eigenvalues lower than one, i.e., x_t is a stationary process.

The system of linear rational expectations (3.1) and the dynamics of the unobserved shocks (3.2) nests several small-scale linearized DSGE models, such as the benchmark New Keynesian model. Note that this system does not allow for endogenous state variables. I do not include them because they complicate the solution method without introducing any insight to this particular problem. Chapter 3 of this thesis proposes a new solution method that includes endogenous state variables and shows that they do not affect any properties higher-order expectations.

For the simple example, the definitions of the endogenous and exogenous shocks are $Y_t = [y_t \quad p_t]'$ and $x_t = [m_t \quad v_t]'$, respectively. The matrices of related to the shocks are given by $A = diag([\rho \quad 0])$, $\Sigma_\varepsilon = diag([\sigma_\varepsilon^2 \quad \sigma_v^2])$.[4] The matrices of the system of equations are such that

$$F = L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 \\ 1-\alpha & 0 \end{bmatrix}, M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.29}$$

[4] $diag(\cdot)$ is the operator creates a new matrix whose diagonal has the elements of the applied matrix, and off-diagonal elements are equal to zero.

In the following, I compare the the model under imperfect information and under imperfect common knowledge. The results from the simple model are generalized to this general setting.

2.4.1 Imperfect information model

In the imperfect (common) information (ICI) model, agents observe only the public signals s_t given by

$$s_t = c_{x,1}x_t + c_y Y_t + d_u u_t, \quad u_t \sim \mathcal{N}(0, \Sigma_u) \tag{2.30}$$

where Σ_u is a $p \times p$ diagonal matrix and s_t is a $p \times 1$ vector. s_t may contain endogenous variables – for instance, prices (see Morris and Shin (2005), Angeletos and Werning (2006), Amador and Weill (2010) and many others) or the nominal interest rate (Erceg and Levin; 2003; Schorfheide; 2005; Andolfatto et al.; 2008; Melosi; 2017) – and exogenous noisy public signals (Angeletos and La'O; 2010; Lorenzoni; 2009; Blanchard et al.; 2013).

In terms of the simple model, the matrices of the public signal are defined as: $c_{x,1} = d_u = 1$, $c_y = [\phi_y \quad \phi_p]$ and $\Sigma_u = \sigma_u$.

As before, since agents form expectations based on a common information, their expectations and optimal choices must be the same. Therefore, for each agent i, $E_{it}[\cdot] = E[\cdot|\mathcal{I}_t]$, where $\mathcal{I}_t$ is the agents' common information. In the same way as in the simple model, since agents have the same decisions, they can infer the aggregate endogenous variables by their own decisions. Thus, agents' information set is defined as $\mathcal{I}_t = \{s_\tau, Y_\tau, \tau \leq t\}$.

This information set has two implications. First, since the endogenous variables are known by agents, the system of rational linear equations becomes

$$FE_t\left[Y_{t+1}\right] + GY_t + L_1 E_t\left[x_{t+1}\right] + L_2 E_t\left[x_t\right] + Mx_t = 0, \tag{2.31}$$

where $G = G_1 + G_2$.

In terms of the simple model, $G = \begin{bmatrix} -1 & -1 \\ 0 & -(1-\alpha) \end{bmatrix}$. This the same argument used in equation (2.1) to show the optimal price decision under imperfect information (equation 2.6).

Second, given their information, agents can extract information from the following *adjusted public signal* computed by

$$\hat{s}_t = s_t - c_y Y_t = c_{x,1}x_t + d_u u_t. \tag{2.32}$$

The equation above simply states that agents use their knowledge of the endogenous variables to find a signal that depend only on exogenous shocks.

One obvious but key assumption is that the adjusted public signals can not perfectly reveal the unobservable shocks. That is, either there are less signals than unobserved shocks ($p < n$) or, if the number of signals is equal or higher to the number of shocks ($p \geq n$), the noises have a sufficiently high variance. In other words, there is a meaningful signal extraction problem.

Agents form expectations about the underlying unobserved shocks (equation 3.2) given information given by equation (2.32). The following proposition shows how rational expectations under imperfect information, $E_t[\cdot] \equiv E[\cdot|\mathcal{I}_t]$, are formed.

Lemma 2.1. *Given the observational equation (2.32), the rational expectation about unobservable shocks, x_t, is given by*

$$E_t(x_t) = (I_n - \bar{K}c_{x,1})A_1E_{t-1}(x_{t-1}) + \bar{K}(c_{x,1}A_1x_{t-1} + c_{x,1}\varepsilon_t + d_u u_t), \tag{2.33}$$

where $\bar{K}$ is the steady-state Kalman gain, which can be computed by solving the Riccati equation that combines the following equations:

$$\begin{aligned} \bar{K} &= \bar{P}c_{x,1}' \left[c_{x,1}\bar{P}c_{x,1}' + d_u\Sigma_u d_u'\right]^{-1} \\ \bar{P} &= A_1(I_n - \bar{K}c_{x,1})\bar{P}A_1' + \Sigma_\varepsilon \end{aligned} \tag{2.34}$$

The proof is a simple application of the Kalman filter and is left in the Appendix 2.B.1. Since the model is linear and the shocks and signals are normally distributed, rational expectations can be computed by the Kalman filter.

Given the dynamics of expectations characterized in Lemma 2.1, the solution of the system of linear rational expectations (2.31) is found. The following Lemma shows the equilibrium conditions.

Lemma 2.2. *The system linear rational expectations equations (2.31) has a solution in the following form*

$$Y_t = Q_0x_t + Q_1E_t[x_t] \tag{2.35}$$

such that the equilibrium matrices, Q_0 and Q_1, and satisfy the matrix equations:

$$GQ_0 + M = 0_{m\times n} \tag{2.36a}$$

$$GQ_1 + F(Q_0 + Q_1)A_1 + (L_1A_1 + L_2) = 0_{m\times n} \tag{2.36b}$$

The proof is in the Appendix 2.B.2. Lemmas 2.1 and 2.2 show how to compute the dynamics of the endogenous variables and expectations under imperfect information. In the following section, I show the same for the imperfect common knowledge case and compare the equilibrium conditions for both models.

2.4.2 Imperfect common knowledge Model

There are two additional features in the imperfect common knowledge (ICK) model. First, in addition to the noisy public signals, each agent $i \in [0,1]$ also observe the private signal given by

$$s_{it} = c_{x,2} x_t + d_w w_{it}, \quad w_{it} \sim \mathcal{N}(0, \Sigma_w) \tag{2.37}$$

where Σ_w is a $q \times q$ diagonal matrix and s_{it} is a $q \times 1$ vector.

Second, I assume that agents do not extract information from idiosyncratic endogenous variables. For the simple model, the timing assumption that agents choose prices before production takes place implies that firms do not use their output decisions to extract information from the money supply.

The private signal induces strategic uncertainty to agents, which break down two key features from imperfect common information models. First, as long as G_2 is not a null matrix, i.e., at least some of the decisions are strategic complements (or substitutes), then higher-order expectations about the unobservable shocks matter for the equilibrium outcome. This can be seen, for instance, by isolating Y_t in the system (3.1), taking the higher-order expectations of Y_t and substituting it back iteratively in system. For instance, this is the procedure used to find equation (2.16) from the equation (2.1).

Second, strategic uncertainty changes the information available to agents. Trivially, agents have an additional information s_{it}, but more interestingly, the fact that agents to not know other agents' decisions prevent them from knowing current aggregate variables, Y_t. Therefore, the information set of each agent i is defined as $\mathcal{I}_t^i = \{s_\tau, s_{i\tau}, Y_{\tau-1}, \tau \leq t\}$.

Note that the endogenous variables are not observed unless they are included in the public signals. In the simple model, for instance, z_t is a endogenous variable that by assumption firms observe whereas y_t and p_t are the endogenous variables that agents are unable to observe owing to the strategic uncertainty and the timing assumption.

Following Nimark (2008), denote the expectations hierarchy $x_t^{(0:\bar{k})} \equiv \left[\begin{array}{cccc} x_t' & E_t^{(1)}[x_t]' & \cdots & E_t^{(\bar{k})}[x_t]' \end{array} \right]'$ the vector that stacks the average higher-order expectations from order 0 to $\bar{k}$. $\bar{k}$ is last the order of average expectation that has impact on the

equilibrium dynamics.[5]

The following Lemma shows the solution of the linear rational expectations model, conditional on the hierarchy of expectations.

Lemma 2.3. *The system of equations (3.1) has the equilibrium law of motion such that:*

$$\begin{aligned} Y_t &= Q x_t^{(0:\bar{k})} \\ x_t^{(0:\bar{k})} &= A x_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t \end{aligned} \tag{2.38}$$

where given the guessed matrices, A, B_ε and B_u, the matrix Q and satisfy the matrix equations:

$$G_1 Q + G_2 Q T + F Q A T + (L_1 A_1 + L_2) e_n' T + M e_n' = 0 \tag{2.39}$$

where T is the transformation matrix such that $E_t^{(1)}[x_t^{(0:\bar{k})}] = T x_t^{(0:\bar{k})}$ and e_n is the selection matrix such that $x_t = e_n' x_t^{(0:\bar{k})}$.

Lemma 3.1 is shown in Chapter 1. The proof is in the Appendix for completeness. This solution method extends the methods from Nimark (2008) and Melosi (2017) to include public signals into agents' information set.

This Lemma shows how the endogenous variables depends on the hierarchy of expectations. Note that the equilibrium Q depends on the guessed persistence (A) of the hierarchy as long as the problem is forward-looking (F is not null).[6] Provided that there is a solution, the matrix Q can be computed by straightforward vectorization.

As Y_t is not included in the information set of agents, those variables cannot be treated as a observable variable in equation (3.3). Instead, agents use their knowledge that the average higher-order expectations drive the endogenous variable dynamics. Therefore, given the guessed solution for Y_t, one can use this result write the observables regarding the expectation hierarchy such that

$$\begin{bmatrix} s_t \\ s_{it} \end{bmatrix} = \begin{bmatrix} c_{x,1} e_n' + c_y Q \\ c_{x,2} e_n' \end{bmatrix} x_t^{(0:\bar{k})} + \begin{bmatrix} d_u & 0_{q\times p} \\ 0_{p\times q} & d_w \end{bmatrix} \begin{bmatrix} u_t \\ w_{it} \end{bmatrix}, \tag{2.40}$$

where $e_n \equiv \begin{bmatrix} I_n & 0_{k-n\times n}' \end{bmatrix}'$ is a $k \times n$ matrix. This can be rewritten simply as

[5] Nimark (2017) shows as impact on equilibrium outcomes of higher-order expectations decreases with the order, that there exists a $\bar{k}$ for such that the approximation error of the solution is lower than any $\epsilon > 0$. As discussed before, for the purposes of this paper, the approximation error is irrelevant in the limiting case because higher-order expectations converge to a common expectation.

[6] If the problem is static, but the learning is still dynamic, Woodford (2002) proposes an exact solution that for the model. Morris and Shin (2002) use a simple model with both static problem and learning to discuss the value of public information.

$$S_{it} = Cx_t^{(0:\bar{k})} + D_u u_t + D_w w_{it}, \tag{2.41}$$

where $S_{it} = \begin{bmatrix} s_t' & s_{it}' \end{bmatrix}'$ and C, D_u, and D_v are defined accordingly. S_{it} is the vector of observed variables that provide information about the unobserved shocks and their higher-order expectations.

Differently from the ICI model, the matrix C depends on the equilibrium response to shocks Q. Therefore, this reflect that the signal extraction problem has to deal with the endogenous response to shocks and their higher-order expectations. Thus, the ICK model allows a interesting feedback loop from the responses of endogenous variables to the expectations formation. This relationship together with the relation between Q and A implies a fixed point problem to solve the model.

Each agent i uses the Kalman filter to form expectations $E_{it}[\cdot] \equiv E[\cdot|\mathcal{I}_t^i]$ about $x_t^{(0:\bar{k})}$ given by the guessed equations (3.7) and use the equation (3.20) as observational equation. Given the individual expectations it is possible to compute the average (first-order) expectation about $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$. The following Lemma shows how those expectations are computed by the Kalman filter.

Lemma 2.4. *Each agent i with the observational equation given by (3.20) and the guessed law of motion of $x_t^{(0:\bar{k})}$ given by (3.7) has the following individual expectation:*

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}\left(CB_u + D_u\right)u_t + \bar{K}D_w w_{it} \tag{2.42}$$

and the average first-order expectation of $x_t^{(0:\bar{k})}$ is given by

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) AE_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}\left(CB_u + D_u\right)u_t \tag{2.43}$$

where $\bar{K}$ is the steady-state Kalman gain is computed by the solution of the Riccati equation given by:

$$\bar{K} = \left[\bar{P}C' + \Sigma_{cov}\right]\left[C\bar{P}C' + \Sigma_{obs}\right]^{-1}, \tag{2.44}$$

and

$$\bar{P} = A\left[\bar{P} - \bar{K}\left(\bar{P}C' + \Sigma_{cov}\right)'\right]A' + \Sigma_{state} \tag{2.45}$$

where $\Sigma_{obs} = D_u\Sigma_u D_u' + D_w\Sigma_w D_w'$, $\Sigma_{state} = B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u'$ and $\Sigma_{cov} = B_u\Sigma_u D_u'$.

Lemma 3.2 is also shown in Chapter 1. This Lemma shows that the individual expectations of higher-order expectations differ only by the different realizations of w_{it} and noise in the public signal u_t is a additional source of variation to the model that affects endogenous variables only through changes in average expectations.

Lorenzoni (2009), and Angeletos and La'O (2010) discuss the importance of those noise shocks about productivity to explain business cycles.[7] Moreover, Chapter 1 uses public signals about the inflation target to evaluate whether greater central bank's transparency enhances the effectiveness of monetary policy. That paper shows that this is the case only under imperfect common knowledge.

Lemma 3.2 also shows that agents must that take into account the correlation of errors in the state equation and the observational equation (note that u_t appears in both equations). This is why the term Σ_{cov} enters in the equations (3.22) and (3.23). Intuitively, agents take into account that public signals also affect the expectations of other agents when forming expectations about higher-order expectations. This changes how one would update the expectations given the observation for both public and private signals, which is given by the Kalman gain $\bar{K}$. This result relates to Morris and Shin (2002), which shows that agents with strategic complementarities tend to overreact to public signals vis-a-vis private signals relatively the reaction expected by their relative precision.

Chapter 1 shows how to find the equilibrium matrices A, B_ε and B_u. By using Lemma 3.2 with some straightforward but cumbersome matrix algebra one can find the guessed law of motion from equation (3.7). I refer the reader that paper for a detailed derivation. For this paper, since I focus in the limiting case the variance of the private signal becomes unbounded large, I show in the following that those equilibrium matrices can be computed in a simpler way.

2.4.3 Imperfect common knowledge and imperfect information: A limiting result

Idiosyncratic signals have a dual role in terms of informativeness. They provide additional information about unobserved variables but also prevent agents from knowing current endogenous variables. In this section, I show that under certain conditions, even in the limiting case that the private signals do not provide any information, the fact that agents do not observe endogenous variables is still matters for equilibrium outcome.

In this section, define the variance of private signals, without loss of generality, as $\Sigma_w =$

[7] Unlike in this paper, those papers use the simplifying assumption that period $t-1$ unobservable shocks become common knowledge at period t.

$\Sigma\sigma_w^2$, where Σ is a diagonal matrix and σ_w is a scalar. Then, I compute the equilibrium the for the limit $\sigma_w \to \infty$ and compare the with the equilibrium conditions of under imperfect information.

Kalman gain matrix given by equation (3.22)can be expressed as

$$\bar{K} = \begin{bmatrix} \bar{K}_u & \bar{K}_w \end{bmatrix}, \tag{2.46}$$

where $\bar{K}_u$ is a $k \times p$ matrix of Kalman gain related to the public signals and $\bar{K}_w$ is a $k \times q$ matrix of the Kalman gain of private signals. Using the definition of C and Σ_{obs} and some matrix algebra, the decomposition of $\bar{K}$ can be computed as

$$\begin{bmatrix} \bar{K}_u & \bar{K}_w \end{bmatrix} = \begin{bmatrix} \bar{P}(c_{x,1}e_n' + c_y Q)' + \Sigma_{cov} & \bar{P}(c_{x,2}e_n')' \end{bmatrix} \begin{bmatrix} (c_{x,1}e_n' + c_y Q)\bar{P}(c_{x,1}e_n' + c_y Q)' + d_u \Sigma_u d_u' & (c_{x,1}e_n' + c_y Q)\bar{P}(c_{x,2}e_n')' \\ (c_{x,2}e_n')\bar{P}(c_{x,1}e_n' + c_y Q)' & (c_{x,2}e_n')\bar{P}(c_{x,2}e_n')' + \sigma_w^2 d_w \Sigma d_w' \end{bmatrix}^{-1}. \tag{2.47}$$

Then, using this decomposition one can rewrite the individual expectation from Lemma 3.2 as

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) A E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon \varepsilon_t + \bar{K}\left(CB_u + D_u\right)u_t + \bar{K}_w d_w w_{it}. \tag{2.48}$$

Therefore, $\bar{K}_w$ controls how individual higher-order expectations responds to the noise of the idiosyncratic signal, which is the only source of disagreement of the expectations.

Using the standard for the block matrix inversion and computing the limit $\sigma_w \to \infty$, one can show that the Kalman gain is given by

$$\bar{K}_u = \left[\bar{P}(c_{x,1}e_n' + c_y Q)' + \hat{\Sigma}_{cov}\right]\left[(c_{x,1}e_n' + c_y Q)\bar{P}(c_{x,1}e_n' + c_y Q)' + d_u \Sigma_u d_u'\right]^{-1} \tag{2.49a}$$

$$\bar{K}_w = 0_{k \times q} \tag{2.49b}$$

where $\hat{\Sigma}_{cov} = B_u \Sigma_u d_u'$. Equation (2.49b) implies that agents to not update based on information from the private signal, x_{it}. Moreover, equation (2.49a) is the Kalman gain related to the public signal. Note that it is computed in the same manner as if only the public signal was the only signal available to agents. In other words, it is equivalent to compute the expectations dynamics through the Kalman filter and applying the limit $\sigma_w \to \infty$ or

computing the Kalman gain considering only the public signal as observable. Therefore, this proves the claim used to solve the simple model in Section 2.3.

Moreover, equations (2.48) and (2.49b) imply that, in the limiting case, individual expectations of average higher-order expectations are equal. Formally, $E_{i,t}(x_t^{(0:\bar{k})}) = E_{j,t}(x_t^{(0:\bar{k})}) = \hat{E}_t(x_t^{(0:\bar{k})})$ for any $i, j \in [0,1]$ such that $\hat{E}_t[\cdot] \equiv \lim_{\sigma_w \to \infty} E[\cdot|x_{it}, z_t]$. Since expectations are common, then higher-order expectations are all equal, i.e., $E_t^{(k)}(x_t) = \hat{E}_t(x_t)$ for any $k \geq 1$. Therefore, by definition, the expectations hierarchy becomes $x_t^{(0:\bar{k})} \equiv \begin{bmatrix} x_t' & \hat{E}_t[x_t]' & \hat{E}_t[x_t]' & \cdots & \hat{E}_t[x_t]' \end{bmatrix}'$, i.e., all orders higher than one become redundant.

Therefore, without loss of generality, I can redefine the guessed solution and signal extraction problem such that: i) the expectations hierarchy account only up to the first-order expectation about the unobservable shocks; ii) agents use the only the public signals as source of information.

The hierarchy of expectations is redefined as

$$\begin{bmatrix} x_t \\ \hat{E}_t[x_t] \end{bmatrix} = \begin{bmatrix} A_1 & 0_{n\times n} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \hat{E}_{t-1}[x_{t-1}] \end{bmatrix} + \begin{bmatrix} I_n \\ b_\varepsilon \end{bmatrix} \varepsilon_t + \begin{bmatrix} 0_{n\times p} \\ b_u \end{bmatrix} u_t. \tag{2.50}$$

Then, matrices A, B_ε, and B_u are also redefined accordingly such that $\hat{X}_t = A\hat{X}_{t-1} + B_\varepsilon \varepsilon_t + B_u u_t$, where $\hat{X}_t = \begin{bmatrix} x_t' & \hat{E}_t[x_t]' \end{bmatrix}'$. Now $A_{21}, A_{22}, b_\varepsilon$ and b_u are guessed coefficient matrices that need to be determined in equilibrium.

Furthermore, the equilibrium response of the endogenous variables is redefined as

$$Y_t = \hat{Q}_0 x_t + \hat{Q}_1 \hat{E}_t[x_t] = \hat{Q}\hat{X}_t, \tag{2.51}$$

where $\hat{Q}_0$ and $\hat{Q}_1$ are also guessed coefficient matrices to be pinned-down in equilibrium.

Using the redefined hierarchy of expectations, the observational equations becomes

$$s_t = \begin{bmatrix} c_{x,1} + c_y\hat{Q}_0 & c_y\hat{Q}_1 \end{bmatrix} \begin{bmatrix} x_t \\ \hat{E}_t[x_t] \end{bmatrix} + d_u u_t = \hat{C}\hat{X}_t + d_u u_t. \tag{2.52}$$

where $\hat{C}$ is defined accordingly.

The signal extraction problem in the case of a non-informative private signal is computed by the Kalman filter using equations (2.50) and (2.52) as state and observational equations, respectively.

The following proposition shows the expectation formation and its equilibrium dynamics in this limiting case.

Proposition 2.3. *Suppose that x_t is a stationary process and consider the case of a*

non-informative private signal ($\sigma_w \to \infty$). *Then, the dynamics of the expectations* $\hat{E}_t(x_t)$ *is given by*

$$\hat{E}_t(x_t) = A_{21}x_{t-1} + A_{22}\hat{E}_{t-1}(x_{t-1}) + b_\varepsilon \varepsilon_t + b_u u_t, \tag{2.53}$$

where the equilibrium coefficients are such that

$$\begin{aligned} A_{21} &= \hat{K}_e \left(c_{x,1} + c_y \hat{Q}_0\right) A_1 \\ A_{22} &= \left(I_n - \hat{K}_e \left(c_{x,1} + c_y \hat{Q}_0\right)\right) A_1 \\ b_\varepsilon &= \hat{K}_e \left(c_{x,1} + c_y \hat{Q}_0\right) \\ b_u &= \hat{K}_e d_u \end{aligned} \tag{2.54}$$

where $\hat{K}_e$ *is the steady-state effective Kalman gain is found by the solution of the Riccati equation given by:*

$$\begin{aligned} \hat{K} &= \left[\hat{P}(c_{x,1} + c_y\hat{Q}_0)'\right] \left[(c_{x,1} + c_y(\hat{Q}_0 + \hat{Q}_1))\hat{P}(c_{x,1} + c_y\hat{Q}_0)' + d_u\Sigma_u d_u'\right]^{-1} \\ \hat{K}_e &= \left[I_n - \hat{K}c_y\hat{Q}_1\right]^{-1}\hat{K} \\ &= \left[\hat{P}(c_{x,1} + c_y\hat{Q}_0)'\right] \left[(c_{x,1} + c_y\hat{Q}_0)\hat{P}(c_{x,1} + c_y\hat{Q}_0)' + d_u\Sigma_u d_u'\right]^{-1} \\ \hat{P} &= A\left[I_n - \hat{K}_e(c_{x,1} + c_y\hat{Q}_0)'\right]\hat{P}A' + \Sigma_\varepsilon \end{aligned} \tag{2.55}$$

where $\hat{K}$ *and* $\hat{P}$ *are the steady-state Kalman gain matrix and the mean square error of the one-period ahead forecast of* x_t*, respectively.*

Then, the expectations of the case of a non-informative private signal are equal to the imperfect information case if and only if:

$$c_y\hat{Q}_0 = 0_{p\times n} \tag{2.56}$$

The proof is in the Appendix 2.B.3. This proposition shows that in the case of a non-informative private signal, the equilibrium dynamics is different from the model with imperfect information owing to the differences in expectations formation.

As shown in the simple model, it is the effective Kalman gain that matters for the expectations dynamics and not the standard Kalman gain. The former disregards, from the standard Kalman gain, the information from the public signal that comes from the expectations itself. This is done by applying the transformation $\left[I_n - \hat{K}c_y\hat{Q}_1\right]^{-1}$ to $\hat{K}$ (see equation (2.55)). The term $\hat{K}c_y\hat{Q}_1$ determines by how much the expectation $\hat{E}_t(x_t)$ responds to variations from s_t that comes from itself.

In the same way as before, the effective Kalman gain is equivalent to the Kalman gain

that would arise if firms observe the effective public signal given by

$$\tilde{s}_t \equiv s_t - c_y \hat{Q}_1 \hat{E}_t(x_t) = (c_{x,1} + c_y \hat{Q}_0) x_t + d_u u_t. \tag{2.57}$$

In the simple model, the velocity shock, v_t was absorbed in the noise of the public signal. Here, instead, that shock is was incorporated within the definition of x_t. Therefore, both the additional response to unobserved shocks and the potential further noise are in the term $c_y \hat{Q}_0 x_t$ in the equation above whereas the term $c_{x,1} x_t$ is the original response from the public signal (see equation 3.3).

The term $c_y \hat{Q}_0 x_t$ comes from the equilibrium the endogenous effect of shocks on the public signal whereas $c_{x,1} x_t$ is the direct effects. Therefore, it is precisely the endogenous effect that is unknown owing to the strategic uncertainty that changes the informational content of the public signal.

Proposition 2.3 states that the equilibrium under imperfect information and the non-informative private signal coincide only, and only if, the condition $c_y \hat{Q}_0 = 0_{p \times n}$ is satisfied. In that case, the effective public signal and the adjusted public signal are the same (compare equations 2.57 and 2.32). Therefore, the effective Kalman gain, $\hat{K}_e$, and the mean square error of the forecast of x_t, $\hat{P}$, would be the same as their counterpart under imperfect information, $\bar{K}$ and $\bar{P}$ (compare equations 2.55 and 2.34). The same is true for the dynamics of expectations, where the coefficients $A_{21}, A_{22}, b_\varepsilon$ and b_u becomes the same as the ones implicit in the expectations formation under imperfect information (compare 2.54 and 2.33).

Therefore, the both models have the same expectation formation if: i) the signal extraction do not include endogenous variables, $c_y = 0_{p \times m}$; ii) the endogenous variables, in equilibrium, do not respond to the shocks, $\hat{Q}_0 = 0_{m \times n}$; or iii) there at least one endogenous variables that respond to shocks but agents to not observe any of those variables, i.e., $c_y \neq 0_{p \times m}$ and $\hat{Q}_0 \neq 0_{m \times n}$ but $c_y \hat{Q}_0 = 0_{p \times n}$ still holds.

Regarding the simple model, recall that the expectation dynamics, in the limiting case, depends on ϕ_y but not on ϕ_p. This is true because output responds directly to money supply changes whereas the price level does not. The latter only depends on the expectations about the money supply, whose information is disregarded by the effective public signal.

Note that the condition (i) is equivalent to assume $\phi_y = 0$, and condition (ii) holds individually for the price level but not for output. The same extreme case that agents do not extract any information from the public signal is also present in the general case. That is the case if the condition $c_y \hat{Q}_0 = -c_{x,1}$ ($\phi_y = -1$ for the simple model). In that case, agents do not update expectations from the public signal ($\hat{K}_e = 0_{n \times p}$) and the expectation is equal to

the unconditional expectation ($\hat{E}[x_t] = 0_{n\times 1}$ with covariance matrix equals to $\hat{P} = \Sigma_\varepsilon$).

In the general case, there is another reason why equilibrium outcomes of the limiting case of the imperfect common knowledge model can differ from the one under imperfect information. Even if expectation formation are the same in both models, it still can be the case that the responses of endogenous variables to shocks and/or expectations about the shocks differ. This is note case in the simple model since the responses are the same.

The system of rational expectations in the limiting case of a non-informative private signal is the same as in equation (3.1) but using the expectation operator $\hat{E}[\cdot]$, such that

$$F\hat{E}_t\left[Y_{t+1}\right] + G_1 Y_t + G_2\hat{E}_t\left[Y_t\right] + L_1\hat{E}_t\left[x_{t+1}\right] + L_2\hat{E}_t\left[x_t\right] + Mx_t = 0_{m\times n}. \tag{2.58}$$

There is two differences from equation (2.58) to the system of equations is the under imperfect information (equation 2.31). First, the expectation operator applied is $\hat{E}_t$. Second, the term $G_2\hat{E}_t[Y_t]$ does not vanish as in the case of imperfect information because the endogenous variables are not observed.

The following Proposition shows how the equilibrium matrices $\hat{Q}_0$ and $\hat{Q}_1$ can be computed and the differences of their counterparts in the imperfect information model.

Proposition 2.4. *Consider the case of a non-informative private signal ($\sigma_w \to \infty$). The matrices of the guessed solution from equation (2.51) for the system of linear rational expectations from equation (2.58) satisfy the following matrix equations:*

$$G_1\hat{Q}_0 + M = 0_{m\times n} \tag{2.59a}$$

$$G\hat{Q}_1 + F(\hat{Q}_0 + \hat{Q}_1)A_1 + G_2\hat{Q}_0 + (L_1A_1 + L_2) = 0_{m\times n}. \tag{2.59b}$$

Then, the equilibrium outcomes of the case of a non-informative private signal are equal to the imperfect information case if and only if:

$$G_2\hat{Q}_0 = 0_{m\times n}. \tag{2.60}$$

The proof is in the Appendix 2.B.4. If the condition stated in equation (2.60) is true, then equilibrium conditions for the limiting case of the ICK model is (equation 2.59) is the same of the conditions for the ICI model (equation 2.36. therefore, it must be the case that $\hat{Q}_0 = Q_0$ and $\hat{Q}_1 = Q_1$.

Therefore, the both models have the same responses to shocks and its expectations if: i) the endogenous variables, in equilibrium, do not respond to the shocks, $\hat{Q}_0 = 0_{m\times n}$; ii) the equilibrium conditions do not include expectations about current endogenous variables , $G_2 = 0_{m\times m}$; or iii) $\hat{Q}_0 \neq 0_{m\times n}$ and $G_2 \neq 0_{m\times m}$ but $G_2\hat{Q}_0 = 0_{p\times n}$ still holds. The latter case happens when, the set of endogenous variables that responds to shocks do not depend on the expectations about current endogenous variables variables and the set of variables that depends on expectations about current endogenous variables are not related to shocks or other variables that depend on those shocks.

For concreteness, consider the simple model presented in Section 2.3. In that model, $p_t = \alpha\hat{E}[p_t] + (1-\alpha)\hat{E}[m_t]$ and $y_t = m_t - p_t + v_t$. Therefore, the price level depends on expectations about itself and about the money supply but does not depend on the money supply (or any endogenous variable that depends on the money supply). Output responds directly to the money supply but do not depend of expectations of any endogenous variables. In that case, condition (iii) is satisfied.

Note that the reaction of endogenous variables to the expectations $\hat{Q}_1$ depends on the persistence of the shock, A_1, and not in the persistence of the expectations, A_{12}. This feature is present with imperfect information but absent under the imperfect common knowledge with a finite σ_w. In the latter, the response to the expectations hierarchy depends on the persistence of that hierarchy (see equation 2.39). This feature is a direct result that the law of iterated expectations holds, which is not the case when there is dispersion in expectations.

This section generalizes the insights provided with the simple model. This discontinuity result from Proposition 2.3 highlights the amount of information about endogenous variables implicitly assumed available to agents in the standard imperfect information models.

Moreover, the results from Proposition 2.3 shows the equilibrium conditions that would arise under imperfect information with the additional assumption that agents cannot infer aggregate outcomes from their own decisions. In other words, agents do not realize that they have the same information. Alternatively, there is no common knowledge that agents have the same information (but they still know that they solve the same problem). Despite that agents do not know other agents' information sets, the law of iterated expectations still holds, since they have the same information, in equilibrium.

I believe that the lack of common knowledge about sharing information is analogous to standard assumptions that the profession (correctly) makes under other circumstances. For instance, when modeling representative households, we assume that this representative agent do not take into account that their decisions affect aggregate outcomes, precisely because it is representation of a individual decision that is aggregated. I advocate here that the same

treatment should be done regarding inferring aggregate decisions whenever agents acquire information from endogenous variables.

2.5 Concluding remarks

This paper compares the signal extraction problems under imperfect information and imperfect common knowledge in the limiting case of a non-informative private signal for a general class of DSGE models. This comparison shows a an interesting discontinuity such that the equilibrium with agents that receive an infinitely noisy private signal does not converge to the equilibrium without this private signal.

The intuition for this is that agents in the imperfect information model infer the aggregate endogenous variables by realizing it has to be the same value as their own decision since all agents are equal. Therefore, this model implicitly assumes that agents know too about other agents, which implies that agents know too much about aggregate endogenous variables. In the limiting case of a non-informative private signal, despite agents decisions are all the same in equilibrium, strategic uncertainty still prevails. That prevents agents from inferring aggregate choices from their own, which in turn, affect the signal extraction problem if agents observe those aggregate endogenous variables.

This suggests that if one is modeling a signal extraction problem from endogenous variables, it is desirable to assume lack of common knowledge about sharing information, i.e., assume that agents do not know that they share the same information set. Otherwise, the agents' expectations formation would be based on more information than individual agents were supposed to have.

2.A Simple model: Proofs

2.A.1 Microfoundations of equation (2.1)

There is a representative household whose utility function is given by

$$U = \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\varphi}}{1+\varphi} \tag{2.A.1}$$

where $0 < \beta < 1$ is the discount factor, γ is the inverse of the intertemporal rate of substitution, and φ is the inverse of the Frisch labor supply elasticity. The composite consumption good, C_t is a Dixit-Stiglitz aggregator $C_t = \left(\int_0^1 (C_{i,t})^{(\varepsilon-1)/\varepsilon} di\right)^{\varepsilon/(\varepsilon-1)}$, where

$C_{i,t}$ is the consumption of each variety i in period t and $\varepsilon > 1$ is the elasticity of substitution between varieties. The household budget constraint is given by

$$P_t C_t = W_t L_t, \tag{2.A.2}$$

where W_t is the nominal wage. There is no saving, but this is not essential for the derivation.

First order conditions imply that

$$\frac{W_t}{P_t} = C_t^{\gamma} L_t^{\varphi}. \tag{2.A.3}$$

$$C_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} C_t, \tag{2.A.4}$$

where the price level of the composite good is given by $P_t = \left(\int_0^1 (P_{i,t})^{(1-\varepsilon)} di\right)^{1/(1-\varepsilon)}$

The first condition is standard labor supply condition such that the real wage is equal to the marginal rate of substitution between consumption and labor. The second is the demand for each variety i at period t.

There is a continuum $i \in [0,1]$ of monopolistic competition firms that produce a differentiated good Y_{it} using a linear production function with labor as its only input such that

$$Y_{it} = L_{it}^{\theta}, \tag{2.A.5}$$

where $\theta \in (0,1]$.

Firms maximize the expected discounted value of their profit, which is given by

$$E_{i,t}\left[P_{it} Y_{it} - W_t L_{it}\right], \tag{2.A.6}$$

subject to the market clearing condition, $Y_{i,t} = C_{i,t}$, the firm individual firm demand (2.A.4) and the production function (2.A.5). The first order condition for prices implies

$$p_{it} = \frac{\varepsilon}{\varepsilon - 1} E_{it}\left[W_t \frac{Y_{it}^{\frac{1}{\alpha}-1}}{\alpha}\right]. \tag{2.A.7}$$

In other words, the price is a mark-up over the expected marginal cost.

Log-linearizing the labor supply, production function, demand for firm i and the optimal price and imposing equilibrium conditions leads to

$$w_t - p_t = \gamma y_t + \varphi l_t \tag{2.A.8}$$

$$y_t = \theta l_t \tag{2.A.9}$$

$$y_{it} = -\varepsilon \left(p_{it} - p_t\right) + y_t \tag{2.A.10}$$

$$p_{it} = E_{it}\left[w_t + \frac{1-\theta}{\theta} y_{it}\right] \tag{2.A.11}$$

By substituting the first three equations into the optimal price decision and rearranging

$$p_{it} = E_{it}\left[p_t + \left(\frac{\theta(\gamma-1)+1+\varphi}{\theta+\varepsilon(1-\theta)}\right) y_t\right] \tag{2.A.12}$$

Assuming the aggregate demand as (2.2) and substituting into the optimal price condition implies that

$$p_{it} = \alpha E_{it}\left[p_t\right] + (1-\alpha) E_{it}\left[m_t\right] \tag{2.A.13}$$

where $\alpha = 1 - \frac{\theta(\gamma-1)+1+\varphi}{\theta+\varepsilon(1-\theta)}$. For standard parametrization ($\theta = 0.3, \gamma = 1, \varphi = 2, \varepsilon = 6$), $\alpha \in [0, 1]$.

The optimal price is a weighted average of the expectations about the price level and the money supply as shown in equation (2.1) in the text.

2.A.2 Proof of Proposition 2.1

For convenience, I restate the state equation, given by

$$m_t = \rho m_{t-1} + \varepsilon_t, \quad \varepsilon_t \tag{2.A.14}$$

and the observational equation

$$\hat{z}_t = m_t + u_t. \tag{2.A.15}$$

Using the Kalman Filter (see, for instance Hamilton; 1995, Chapter 13), it is easy to show that

$$E_t[m_t] = E_{t-1}[m_t] + \bar{k}(\hat{z}_t - E_{t-1}[\hat{z}_t]) \tag{2.A.16}$$

where k is the steady-state Kalman gain. Using that $E_{t-1}[m_t] = \rho E_{t-1}[m_{t-1}]$ and substituting equations (2.A.15) and (2.A.14):

$$E_t[m_t] = (1-\bar{k})\rho E_{t-1}[m_{t-1}] + \bar{k}\rho m_{t-1} + \bar{k}(\varepsilon_t + u_t) \tag{2.A.17}$$

The steady-state Kalman gain solves the following fixed point:

$$\begin{aligned} \bar{k} &= \frac{\bar{p}}{\bar{p} + \sigma_u^2} \\ \bar{p} &= \rho^2(1-\bar{k})\bar{p} + \sigma_\varepsilon^2 \end{aligned} \tag{2.A.18}$$

2.A.3 Proof of Proposition 2.2

First, guess (and verify) the expectation about the money supply, $\hat{E}[m_t]$, to have the following dynamics

$$\hat{E}_t[m_t] = \rho_1 m_{t-1} + \rho_2 \hat{E}_{t-1}[m_{t-1}] + b_\varepsilon \varepsilon_t + b_\xi \xi_t \tag{2.A.19}$$

where ρ_1, ρ_2, b_1 and b_u are coefficients to be determined in equilibrium. Therefore, the state equation that firms use to form expectations is given by

$$M_t = AM_{t-1} + B_\varepsilon \varepsilon_t + B_\xi \xi_t \tag{2.A.20}$$

where $M_t = \begin{bmatrix} m_t & \hat{E}[m_t] \end{bmatrix}'$ and the matrices are such $A = \begin{bmatrix} \rho & 0 \\ \rho_1 & \rho_2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ b_1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$.

Moreover, the observational equation (2.20) can be written in matricial form as

$$z_t = \Phi M_t + \xi_t \tag{2.A.21}$$

where $\Phi \equiv \begin{bmatrix} (1+\phi_y) & (\phi_p - \phi_y) \end{bmatrix}$.

By applying the Kalman filter, one can find that

$$\hat{E}_t[M_t] = \hat{E}_{t-1}[M_t] + \hat{K}(z_t - \hat{E}_{t-1}[z_t]) \tag{2.A.22}$$

where $\hat{K} = \begin{bmatrix} \hat{k}_1 & \hat{k}_2 \end{bmatrix}'$ is the Kalman gain still to be defined. Using the fact that $\hat{E}_t[m_t] = (1 \quad 0)\hat{E}_t[M_t]$, one can find that

$$\hat{E}_t[m_t] = \hat{E}_{t-1}[m_t] + \hat{k}_1(z_t - \hat{E}_{t-1}[z_t]) \tag{2.A.23}$$

From equation (2.20), it is easy to see that $\hat{E}_{t-1}[z_t] = (1+\phi_p)\hat{E}_{t-1}[m_t]$. Substituting this

and equation (2.20) into previous equation, leads to

$$\hat{E}_t[m_t] = (1 - \hat{k}_1(1+\phi_p))\hat{E}_{t-1}[m_t] + \hat{k}_1((1+\phi_y)m_t + (\phi_p - \phi_y)\hat{E}_t[m_t] + \xi_t) \tag{2.A.24}$$

By isolating $E_t[m_t]$, using that $\hat{E}_{t-1}[m_t] = \rho\hat{E}_{t-1}[m_{t-1}]$ and, some algebra, one can find that

$$\hat{E}_t[m_t] = (1 - \hat{k}_e(1+\phi_y))\rho\hat{E}_{t-1}[m_{t-1}] + \hat{k}_e(1+\phi_y)(\rho m_{t-1} + \varepsilon_t) + \hat{k}_e\xi_t \tag{2.A.25}$$

where $\hat{k}_e = \frac{\hat{k}_1}{1-(\phi_p-\phi_y)\hat{k}_1}$ is the adjusted Kalman gain. It adjusts the Kalman gain $\hat{k}_1$ to take into account the response of the public signal to the expectation of m_t. In the text, I discuss in detail this point.

Therefore, from equation (2.A.19), the equilibrium values for the guessed parameters are

$$\begin{aligned} \rho_1 &= \hat{k}_e(1+\phi_y)\rho; \\ \rho_2 &= [1 - \hat{k}_e(1+\phi_y)]\rho; \\ b_\varepsilon &= \hat{k}_e(1+\phi_y); \\ b_\xi &= \hat{k}_e. \end{aligned} \tag{2.A.26}$$

Note that the LIE holds, i.e., $\rho_1 + \rho_2 = \rho$.

The guessed coefficients still depends on the unknown Kalman gain. The Kalman gain is given by

$$\hat{K} = (\hat{P}\Phi' + B_\xi\sigma_\xi^2)(\Phi\hat{P}\Phi' + \sigma_\xi^2)^{-1} \tag{2.A.27}$$

where $\hat{P}$ is the steady-state mean squared error of the one-steap ahead forecast of the state variable, i.e., $\hat{P} = \hat{E}[(M_{t+1} - \hat{E}_t M_{t+1})(M_{t+1} - \hat{E}_t M_{t+1})']$, which is computed by

$$\hat{P} = A\tilde{P}A' + \sigma_\varepsilon^2 B_\varepsilon B_\varepsilon' + \sigma_\xi^2 B_\xi B_\xi' \tag{2.A.28}$$

where $\tilde{P}$ is the steady-state mean squared error of the filtered value of the state variable, i.e., $\tilde{P} = \hat{E}[(M_t - \hat{E}_t M_t)(M_t - \hat{E}_t M_t)']$, which is given by

$$\tilde{P} = \hat{P} - \hat{K}(\hat{P}\Phi' + B_\xi\sigma_\xi^2)' \tag{2.A.29}$$

Since the LIE holds ($\hat{E}_t[m_t] = \hat{E}_t[\hat{E}_t[m_t]]$) and by the definition of M_t and $\tilde{P}$, it is easy to see that

$$\tilde{P} = \begin{bmatrix} \tilde{p} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.A.30}$$

Using equations (2.A.29) and (2.A.30), and some matricial algebra, results to

$$\begin{aligned} \tilde{p} &= \hat{p}_{11} - \hat{k}_1[(1+\phi_y)\hat{p}_{11} + (\phi_p - \phi_y)\hat{p}_{12}] \\ \hat{p}_{12} &= \hat{k}_1[(1+\phi_y)\hat{p}_{21} + (\phi_p - \phi_y)\hat{p}_{22} + b_\xi \sigma_\xi^2] \\ \hat{p}_{21} &= \hat{k}_2[(1+\phi_y)\hat{p}_{11} + (\phi_p - \phi_y)\hat{p}_{12}] \\ \hat{p}_{22} &= \hat{k}_2[(1+\phi_y)\hat{p}_{21} + (\phi_p - \phi_y)\hat{p}_{22} + b_\xi \sigma_\xi^2] \end{aligned} \tag{2.A.31}$$

Moreover, using equations (2.A.28) and (2.A.30) one can find that

$$\begin{aligned} \hat{p}_{11} &= \rho^2 \tilde{p} + \sigma_\varepsilon^2 \\ \hat{p}_{12} &= \hat{p}_{21} = \rho \rho_1 \tilde{p} + b_\varepsilon \sigma_\varepsilon^2 \\ \hat{p}_{22} &= \rho_2^2 \tilde{p} + b_\varepsilon^2 \sigma_\varepsilon^2 + b_\xi^2 \sigma_\xi^2 \end{aligned} \tag{2.A.32}$$

where $\hat{p}_{ij}$ is the (i,j)-th element of the matrix $\hat{P}$.

Using the equilibrium ρ_1 and b_ε from equation (2.A.26), $\hat{p}_{12}$ can be rewritten as

$$\begin{aligned} \hat{p}_{12} &= (1+\phi_y)\hat{k}_e(\rho^2 \tilde{p} + \sigma_\varepsilon^2) \\ &= (1+\phi_y)\hat{k}_e p_{11} \end{aligned} \tag{2.A.33}$$

where the last equality come from first equation of (2.A.32). This relation is key to our result. Substituting back this equation into the first equation of (2.A.31) and rearranging, results to

$$\begin{aligned} \tilde{p} &= \hat{p}_{11} - \hat{k}_1(1+\phi_y)[1 + (\phi_p - \phi_y)\hat{k}_e]\hat{p}_{11} \\ &= \hat{p}_{11} - \left[\frac{\hat{k}_1(1+\phi_y)}{1 - (\phi_p - \phi_y)\hat{k}_1}\right]\hat{p}_{11} \\ &= \hat{p}_{11}\left[1 - (1+\phi_y)\hat{k}_e\right] \end{aligned} \tag{2.A.34}$$

where the second equality and third equalities use the definition of $\hat{k}_e$. Using this and the first equation of (2.A.32)

$$\hat{p}_{11} = \rho^2\left[1 - (1+\phi_y)\hat{k}_e\right]\hat{p}_{11} + \sigma_\varepsilon^2 \tag{2.A.35}$$

To finish the proof, I need to find a relationship of $\hat{k}_e$ and $\hat{p}_{11}$. Using into the third equation

of (2.A.32) and $\hat{p}_{12} = \hat{p}_{21}$ such that

$$\hat{p}_{12} = \frac{\hat{k}_2(1+\phi_y)}{1-(\phi_p-\phi_y)\hat{k}_2}\hat{p}_{11}. \tag{2.A.36}$$

Using this, the equation (2.A.33) and the definition of $\hat{k}_e$, it is easy to see that $\hat{k}_1 = \hat{k}_2$. Then, in the following, I drop de index and denote them as $\hat{k}$. This implies that firms update their expectations about m_t and their expectations about their expectations in the same way when they observe the public signal. Since the LIE holds, the Kalman gains have be the same.

Finally, using the equilibrium b_ε from equation (2.A.26) and equation (2.A.33) into the second equation of (2.A.32) and with some algebra, one can find that

$$\hat{k} = \frac{(1+\phi_y)\hat{p}_{11}}{(1+\phi_y)(1+\phi_p)\hat{p}_{11}+\sigma_\xi^2}. \tag{2.A.37}$$

Then, using this into the definition of $\hat{k}_e$, one can see that

$$\hat{k}_e = \frac{(1+\phi_y)\hat{p}_{11}}{(1+\phi_y)^2\hat{p}_{11}+\sigma_\xi^2}. \tag{2.A.38}$$

Note that the equilibrium guessed coefficients depends only on ρ, ϕ_y and $\hat{k}_e$. Then, the adjusted Kalman gain can be found by the fixed point solution of equations (2.A.38) and (2.A.35).

2.B General model: Proofs

2.B.1 Proof of Lemma 2.1

For convenience, I restate the state equation, given by

$$x_t = A_1 x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \Sigma_\varepsilon) \tag{2.B.39}$$

and the observational equation

$$\hat{s}_t = c_{x,1} x_t + d_u u_t. \tag{2.B.40}$$

Using the Kalman Filter (see, for instance Hamilton; 1995, Chapter 13), it is easy to show that

$$E_t[x_t] = E_{t-1}[x_t] + K_t(\hat{s}_t - E_{t-1}[\hat{s}_t]) \tag{2.B.41}$$

where the Kalman gain, K_t, and mean square error of the one-step ahead prediction error, $P_{t+1|t}$, have the following dynamics:

$$\begin{aligned} K_t &= P_{t|t-1}c'_{x,1}\left[c_{x,1}P_{t|t-1}c'_{x,1} + d_u\Sigma_u d'_u\right]^{-1} \\ P_{t+1|t} &= A_1(I_n - K_t c_{x,1})P_{t|t-1}A'_1 + \Sigma_\varepsilon. \end{aligned} \tag{2.B.42}$$

Combining this equations leads to the well-known Riccati equation

$$P_{t+1|t} = A_1\left[I_n - \left(P_{t|t-1}c'_{x,1}\left[c_{x,1}P_{t|t-1}c'_{x,1} + d_u\Sigma_u d'_u\right]^{-1}\right)c_{x,1}\right]P_{t|t-1}A'_1 + \Sigma_\varepsilon \tag{2.B.43}$$

I follow the literature by focusing in the stationary equilibrium. Therefore, the expectation in the stationary equilibrium is the one which the MSE is in steady-state, i.e., firms update their forecast based on the steady-state Kalman gain. Since x_t is stationary and Σ_ε is positive definite, then there exists a steady-state solution such that $\bar{P} = P_{t+1|t} = P_{t|t-1}$ which implies the steady-state Kalman gain $\bar{K}$ (see Hamilton; 1995, chap. 13). Therefore, taking expectations of $\hat{s}_t$ and using the fact that $E_{t-1}(x_t) = A_1 E_{t-1}(x_{t-1})$, the dynamics of $E_t(x_t)$ can be written as equation as

$$E_t[x_t] = (I_n - \bar{K}c_{x,1})A_1 E_{t-1}[x_{t-1}] + \bar{K}(c_{x,1}A_1 x_{t-1} + c_{x,1}\varepsilon_t + d_u u_t) \tag{2.B.44}$$

Finally, substituting the MSE by its stationary version, $\bar{P}$, into equation (2.B.42) results to

$$\begin{aligned} \bar{K} &= \bar{P}c'_{x,1}\left[c_{x,1}\bar{P}c'_{x,1} + d_u\Sigma_u d'_u\right]^{-1} \\ \bar{P} &= A_1(I_n - \bar{K}c_{x,1})\bar{P}A'_1 + \Sigma_\varepsilon \end{aligned} \tag{2.B.45}$$

□

2.B.2 Proof of Lemma 2.2

For convenience, I restate the system of equations under imperfect information, given by

$$FE_t\left[Y_{t+1}\right] + GY_t + L_1 E_t\left[x_{t+1}\right] + L_2 E_t\left[x_t\right] + Mx_t = 0, \tag{2.B.46}$$

As in the text, guess that the solution of the system (2.B.46) has the form

$$Y_t = Q_0 x_t + Q_1 E_t[x_t] \tag{2.B.47}$$

By computing the expectation of equation (2.B.47), leads to

$$\begin{aligned} E_t^{(1)}\left[Y_{t+1}\right] &= Q_0 E_t[x_{t+1}] + Q_1 E_t\left[E_{t+1}[x_{t+1}]\right] \\ &= \left(Q_0 + Q_1\right) A_1 E_t[x_t] \end{aligned} \tag{2.B.48}$$

where last equality uses equation $E_t[x_{t+1}] = A_1 E_t[x_t]$ and the LIE. Substituting the guessed solution (2.B.47) and the expectations (3.16) into the system of equations (2.B.46) one can find:

$$(GQ_0 + M)x_t + \left[F(Q_0 + Q_1)A_1 + GQ_1 + (L_1 A_1 + L_2)\right] E_t[x_t] = 0_{m\times n}. \tag{2.B.49}$$

where it was used that $E_t[x_{t+1}] = A_1 E_t[x_t]$. This condition must hold for all realizations of x_t and $E_t[x_t]$. Therefore, both coefficients between square brackets must be zero, which leads to conditions (2.36). □

2.B.3 Proof of Proposition 2.3

For convenience, I restate the state equation, given by

$$\hat{X}_t = \hat{A}\hat{X}_{t-1} + \hat{B}_\varepsilon \varepsilon_t + \hat{B}_u u_t, \tag{2.B.50}$$

where $\hat{X}_t = \begin{bmatrix} x_t' & \hat{E}[x_t]' \end{bmatrix}'$, $\hat{A} = \begin{bmatrix} A_1 & 0_{n\times n} \\ A_{12} & A_{22} \end{bmatrix}$, $\hat{B}_\varepsilon = \begin{bmatrix} I_n & b_\varepsilon' \end{bmatrix}'$ and $\hat{X}_t = \begin{bmatrix} 0_{n\times p}' & b_u' \end{bmatrix}'$. Recall that A_{12}, A_{22}, b_ε and b_u are guessed matrices to be determined in equilibrium. Also, the observational equation is given by

$$s_t = \hat{C}\hat{X}_t + d_u u_t, \tag{2.B.51}$$

where $\hat{C} = \begin{bmatrix} c_{x,1} + c_y\hat{Q}_0 & c_y\hat{Q}_1 \end{bmatrix}$.

By applying the Kalman filter, the expectation about $\hat{X}_t$ is given by

$$\hat{E}_t\left[\hat{X}_t\right] = \hat{E}_{t-1}\left[\hat{X}_t\right] + \hat{K}_t(s_t - \hat{E}_{t-1}\left[s_t\right]) \tag{2.B.52}$$

where $\hat{K} = \begin{bmatrix} \hat{K}_1' & \hat{K}_2' \end{bmatrix}'$ is the Kalman gain still to be defined. $\hat{K}_1$ and $\hat{K}_2$, are $n \times p$ matrices of the Kalman gains with regard to x_t and $\hat{E}_t x_t$, respectively.

Using the fact that $\hat{E}_t[x_t] = (I_n \quad 0_{n\times n})\hat{E}_t[\hat{X}_t]$, one can find that

$$\hat{E}_t[x_t] = \hat{E}_{t-1}[x_t] + \hat{k}_1(s_t - \hat{E}_{t-1}[s_t]) \tag{2.B.53}$$

Using the fact that the Law of interated expectations (LIE) hold and the definition of $\hat{C}$, by taking the expectation of equation (2.B.51), one can see that $\hat{E}_{t-1}[s_t] = \left(c_{x,1} + c_y(\hat{Q}_0 + \hat{Q}_1)\right)\hat{E}_{t-1}[x_t]$. Substituting this and equation (2.B.51) into previous equation leads to

$$\hat{E}_t[x_t] = \left[I_n - \hat{K}_1\left(c_{x,1} + c_y(\hat{Q}_0 + \hat{Q}_1)\right)\right]\hat{E}_{t-1}[x_t] + \hat{K}_1\left[\left(c_{x,1} + c_y\hat{Q}_0\right)x_t + c_y\hat{Q}_1\hat{E}_t[x_t] + d_u u_t\right] \tag{2.B.54}$$

By isolating $\hat{E}_t[x_t]$, using that $\hat{E}_{t-1}[x_t] = A_1\hat{E}_{t-1}[x_{t-1}]$ and some algebra, results to

$$\hat{E}_t[x_t] = \left[I_n - \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)\right]A_1\hat{E}_{t-1}[x_{t-1}] + \hat{K}_e\left[\left(c_{x,1} + c_y\hat{Q}_0\right)(A_1x_{t-1} + \varepsilon_t) + d_u u_t\right] \tag{2.B.55}$$

where $\hat{K}_e = \left(I_n - \hat{K}_1 c_y\hat{Q}_1\right)^{-1}\hat{K}_1$ is the effective Kalman gain.[8] It adjusts the Kalman gain $\hat{K}_1$ to take into account the response of the public signal to the expectation of x_t. In the text, I discuss this point in detail.

Therefore, the equilibrium values for the guessed parameters are

$$\begin{aligned} A_{12} &= \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)A_1; \\ A_{22} &= \left[I_n - \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)\right]A_1; \\ b_\varepsilon &= \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right); \\ b_u &= \hat{K}_e d_u. \end{aligned} \tag{2.B.56}$$

Note that A_{12} and A_{22} are consistent with the LIE, i.e., $A_{12} + A_{22} = A_1$. The guessed coefficients still depend on the unknown effective Kalman gain, $\hat{K}_e$, and its corresponding mean squared error.

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To find that, note that the steady-state mean squared error of the one-steap ahead forecast of the state variable, $\hat{P}_1 = \hat{E}[(X_{t+1} - \hat{E}_t X_{t+1})(X_{t+1} - \hat{E}_t X_{t+1})']$, can computed by

$$\hat{P}_1 = \hat{A}\hat{P}_0\hat{A}' + B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u' \tag{2.B.57}$$

where $\hat{P}_0$ is the steady-state mean squared error of the filtered value of the state variable, i.e., $\hat{P}_0 = \hat{E}[(X_t - \hat{E}_t X_t)(X_t - \hat{E}_t X_t)']$, which is given by

[8] This expression uses the fact that $\left(I_n - \hat{K}_1 c_y\hat{Q}_1\right)^{-1}\left[I_n - \hat{K}_1\left(c_{x,1} + c_y(\hat{Q}_0 + \hat{Q}_1)\right)\right] = I_n - \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)$. Using the definition of $\hat{K}_e$ and some algebra one can find this equality.

$$\hat{P}_0 = \hat{P}_1 - \hat{K}(\hat{P}\hat{C}' + B_u\Sigma_u d_u')' \tag{2.B.58}$$

Since the LIE holds ($\hat{E}_t[x_t] = \hat{E}_t[\hat{E}_t[x_t]]$) and by the definition of $\hat{X}_t$ and $\hat{P}_0$, it is easy to see that

$$\hat{P}_0 = \begin{bmatrix} \hat{p}_0 & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \tag{2.B.59}$$

Using equations (2.B.58) and (2.B.59), and some matricial algebra, results to

$$\begin{aligned}
\tilde{p} &= \hat{P}_{11} - \hat{k}_1[\left(c_{x,1} + c_y\hat{Q}_0\right)\hat{P}_{11} + c_y\hat{Q}_1\hat{P}_{12}] \\
\hat{P}_{12} &= \hat{k}_1[\left(c_{x,1} + c_y\hat{Q}_0\right)\hat{P}_{12} + c_y\hat{Q}_1\hat{P}_{22} + d_u\Sigma_u b_u'] \\
\hat{P}_{21} &= \hat{k}_2[\left(c_{x,1} + c_y\hat{Q}_0\right)\hat{P}_{11} + c_y\hat{Q}_1\hat{P}_{21}] \\
\hat{P}_{22} &= \hat{k}_2[\left(c_{x,1} + c_y\hat{Q}_0\right)\hat{P}_{12} + c_y\hat{Q}_1\hat{P}_{22} + d_u\Sigma_u b_u']
\end{aligned} \tag{2.B.60}$$

where $\hat{P}_{ij}$ is the (i,j)-th $n \times n$ submatrix of the matrix $\hat{P}_1$.

Moreover, using equations (2.B.57), (2.B.59) and the definition of $(\hat{A}, \hat{B}_\varepsilon, \hat{B}_u)$ leads to

$$\begin{aligned}
\hat{P}_{11} &= A_1\tilde{p}A_1' + \Sigma_\varepsilon \\
\hat{P}_{12} &= \hat{P}_{21}' = A_1\tilde{p}A_{21}' + \Sigma_\varepsilon b_\varepsilon' \\
\hat{P}_{22} &= A_{21}\tilde{p}A_{21}' + b_\varepsilon\Sigma_\varepsilon b_\varepsilon' + b_u\Sigma_u b_u'
\end{aligned} \tag{2.B.61}$$

Using the equilibrium A_{21} and b_ε from equation (2.A.26), $\hat{P}_{21}$ can be rewritten as

$$\begin{aligned}
\hat{P}_{12} &= \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)\left(A_1\tilde{p}A_1' + \Sigma_\varepsilon\right) \\
&= \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)P_{11}
\end{aligned} \tag{2.B.62}$$

where the last equality come from first equation of (2.B.61). This relation is key to our result. Substituting back this equation into the first equation of (2.B.60) and rearranging, results to

$$\begin{aligned}
\tilde{p} &= \hat{P}_{11} - \hat{K}_1\left[\left(c_{x,1} + c_y\hat{Q}_0\right) + c_y\hat{Q}_1\hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)\right]\hat{P}_{11} \\
&= \hat{P}_{11} - \hat{K}_1\left[\left(I_n + c_y\hat{Q}_1\hat{K}_e\right)\left(c_{x,1} + c_y\hat{Q}_0\right)\right] \\
&= \left(I_n - \hat{K}_e\left(c_{x,1} + c_y\hat{Q}_0\right)\right)\hat{P}_{11}
\end{aligned} \tag{2.B.63}$$

where the third equalities uses the definition of $\hat{K}_e$ with some algebra. Using the equation above and the first equation of (2.B.61) leads to

$$\hat{P}_{11} = A_1 \left[\left(I_n - \hat{K}_e \left(c_{x,1} + c_y \hat{Q}_0 \right) \right) \hat{P}_{11} \right] A_1' + \Sigma_\varepsilon \tag{2.B.64}$$

To finish the proof, I need to find a relationship of $\hat{K}_e$ and $\hat{P}_{11}$. Isolating $hat P_{21}$ in the third equation of (2.B.60) leads to

$$\hat{P}_{21} = \left(I_n - \hat{K}_2 c_y \hat{Q}_1 \right)^{-1} \hat{K}_2 \left(c_{x,1} + c_y \hat{Q}_0 \right) \hat{P}_{11} \tag{2.B.65}$$

By comparing equation above with equation (2.B.62) and using the definition of $\hat{K}_e$, it is easy to see that $\hat{K}_1 = \hat{K}_2$. Then, in the following I drop the index and denote them as $\hat{K}$. This implies that agents update expectations about x_t and expectations about expectations in the same way. In other words, the Kalman gains are consistent with the LIE. This implies that $\hat{P}_{12} = \hat{P}_{22}$. Therefore, to simplify the notation, in the following I denote $\hat{P}_{11}$ simply as $\hat{P}$.

Moreover, using the equilibrium b_u from equation (2.A.26) and equation (2.B.62) into the second equation of (2.B.60), one can find that

$$\left[I_n - \hat{K} \left(c_{x,1} + c_y (\hat{Q}_0 + \hat{Q}_1) \right) \right] \hat{P} \left(c_{x,1} + c_y \hat{Q}_0 \right)' \hat{K}_e' = \hat{K} d_u \Sigma_u d_u' \hat{K}_e' \tag{2.B.66}$$

by post-multiplying both sides by $\hat{K}_e (\hat{K}_e' \hat{K}_e)^{-1}$ to eliminate $\hat{K}_e$ from the equation and then isolating $\hat{K}$, results to

$$\hat{K} = \hat{P} \left(c_{x,1} + c_y \hat{Q}_0 \right)' \left[\left(c_{x,1} + c_y (\hat{Q}_0 + \hat{Q}_1) \right) \hat{P} \left(c_{x,1} + c_y \hat{Q}_0 \right)' + d_u \Sigma_u d_u' \right] \tag{2.B.67}$$

By pre-multiplying equation (2.B.66) by $\left[I_n - \hat{K} c_y \hat{Q}_1 \right]^{-1}$ and some algebra, one can find

$$\left[I_n - \hat{K}_e \left(c_{x,1} + c_y \hat{Q}_0 \right) \right] \hat{P} \left(c_{x,1} + c_y \hat{Q}_0 \right)' \hat{K}_e' = \hat{K}_e d_u \Sigma_u d_u' \hat{K}_e' \tag{2.B.68}$$

Finally, by post-multiplying both sides by $\hat{K}_e (\hat{K}_e' \hat{K}_e)^{-1}$ to eliminate some of the $\hat{K}_e$ terms from the equation and then isolating the remaining $\hat{K}_e$, results to

$$\hat{K}_e = \hat{P} \left(c_{x,1} + c_y \hat{Q}_0 \right)' \left[\left(c_{x,1} + c_y \hat{Q}_0 \right) \hat{P} \left(c_{x,1} + c_y \hat{Q}_0 \right)' + d_u \Sigma_u d_u' \right] \tag{2.B.69}$$

Therefore, equilibrium coefficients (2.B.56), the Kalman gains (2.B.67 and 2.B.69) and the mean squared error from (2.B.64) are the ones stated in Proposition 2.3. □

2.B.4 Proof of Proposition 2.4

For convenience, I restate the system of rational expectations in the limiting case, given by

$$FE_t\left[Y_{t+1}\right] + G_1 Y_t + G_2 E_t\left[Y_t\right] + L_1 E_t\left[x_{t+1}\right] + L_2 E_t\left[x_t\right] + M x_t = 0, \tag{2.B.70}$$

As in the text, guess that the solution of the system (2.B.70) has the form

$$Y_t = \hat{Q}_0 x_t + \hat{Q}_1 \hat{E}_t[x_t] \tag{2.B.71}$$

By computing the expectation of equation (2.B.71), one can find that

$$\hat{E}_t^{(1)}\left[Y_t\right] = \left(\hat{Q}_0 + \hat{Q}_1\right) \hat{E}_t[x_t] \tag{2.B.72}$$

where the equality uses the law of iterated expectations (LIE). Similarly, the current expectation for endogenous variables at period $t+1$:

$$\begin{aligned} \hat{E}_t^{(1)}\left[Y_{t+1}\right] &= \hat{Q}_0 \hat{E}_t[x_{t+1}] + \hat{Q}_1 \hat{E}_t\left[\hat{E}_{t+1}[x_{t+1}]\right] \\ &= \left(\hat{Q}_0 + \hat{Q}_1\right) A_1 \hat{E}_t[x_t] \end{aligned} \tag{2.B.73}$$

where last equality uses (2.53) at period $t+1$ and the fact that $A_{12} + A_{22} = A_1$ (see equation 2.54). Substituting the guessed solution (2.B.71) and the expectations (2.B.72-2.B.73) into the system of equations (2.B.70) one can find:

$$(G_1\hat{Q}_0 + M)x_t + \left[F(\hat{Q}_0 + \hat{Q}_1)A_1 + G_1\hat{Q}_0 + G\hat{Q}_1 + (L_1A_1 + L_2)\right] \hat{E}_t[x_t] = 0_{m\times n}. \tag{2.B.74}$$

where $G = G_1 + G_2$. This condition must hold for all realizations of x_t and $\hat{E}_t[x_t]$. Therefore, both coefficients between square brackets must be zero, which leads to conditions (2.59).

Note that if the $G_2\hat{Q}_0 = 0_{m\times n}$ is satisfied, the conditions (2.59) become

$$G_1\hat{Q}_0 + M = 0_{m\times n} \tag{2.B.75a}$$

$$G_1 Q_1 + F(\hat{Q}_0 + \hat{Q}_1)A_1 + (L_1A_1 + L_2) = 0_{m\times n} \tag{2.B.75b}$$

Morever, note that $G\hat{Q}_0 = G_1\hat{Q}_0$, since $G = G_1 + G_2$. Therefore, conditions for the

imperfect information model (equation 2.36) and for the ICK model in the limiting case (2.B.75) are the same. Thus, the response to shocks and expectations about those shocks are the same for both models, i.e., $Q_0 = \hat{Q}_0$ and $Q_1 = \hat{Q}_1$. □

Chapter 3

Solution of linear rational expectations models with imperfect common knowledge

Abstract

This paper develops a novel solution method for a general class of DSGE models with imperfect common knowledge. The main contribution is that the method allows for the inclusion of endogenous state variables into the system of linear rational expectations conditions under imperfect common knowledge. The method also allows for a rich set of exogenous noisy public and private signals and current and lagged endogenous variables into the agents' information sets. One key implication is that the endogenous persistence of state variables is the same under full information and imperfect common knowledge.

This method offers a tractable laboratory that pushes the quantitative frontier for imperfect and dispersed information models by allowing to introduce such informational structure into medium-scale DSGE models. A primer empirical evaluation of the informational frictions reveals that the model under imperfect and dispersed information explains better the expectation data than the full information model. However, this comes at the cost of being relatively worse at explaining the macroeconomic aggregates.

Keywords: DSGE model, solution method, imperfect common knowledge, strategic uncertainty
JEL classification: E3, E32, E37, D83, D84.

3.1 Introduction

The quantitative literature of medium-scale DSGE models following the seminal papers of Christiano et al. (2005) and Smets and Wouters (2007) relies on a variety of reduced-form frictions to better explain the data. However, frictions such as habits, investment adjustment costs, partial indexation are intuitive but lack of substantial empirical evidence from other sources to corroborate those assumptions. Those frictions generate hump-shaped impulse response functions that are consistent with VAR evidence.

On another front, a recent literature starting with Mankiw and Reis (2002), Woodford (2002) and Sims (2003) renewed the interest in using information frictions in macroeconomic models. One key implication of the literature following those papers is that informational frictions induce inertia to the endogenous variables, which can help to explain the dynamics of the key macroeconomic aggregates.

Moreover, gradual learning induced by informational frictions is consistent with key features of survey expectation data. For instance, Andolfatto et al. (2008) document that imperfect information generates bias and persistence in the inflation forecast errors as in the Survey of Professional Forecasters (SFP) data. Coibion and Gorodnichenko (2012) show that a broad class of informational frictions has a common implication: the response of the average forecast error is autocorrelated and has the same sign as the forecasted variable. Using the SPF data, the authors compare the different testable implications from those models and argue that the noisy information models are likely to be the best characterization of the expectations formation.

Despite this renewed interest, there are still a small number of papers providing empirical and quantitative assessments of the relevance of the information frictions for explaining both expectation and macroeconomic data. As discussed by Angeletos and Lian (2016), existing solution methods still have limited applicability, in particular when linking model to the data.

This paper has two main contributions that seek to fill this gap. First, there is a methodological contribution by proposing a solution method for a general class of DSGE models with imperfect and dispersed information. Then, I use the proposed solution to evaluate the role of informational frictions in comparison to standard real and nominal frictions widely used in full information models.

The main contribution of the solution is the inclusion of endogenous state variables into the system of linear rational expectations equations with agents with imperfect common knowledge about the underlying shocks hitting the economy. For concreteness, the imperfect common knowledge assumption is that agents receive noisy idiosyncratic signals about those shocks. Therefore, noisy information, imperfect and dispersed information and imperfect

common knowledge are used interchangeably in the literature. The method enables not only those idiosyncratic signals but also a rich set of observations such as exogenous noisy public signals, lagged and current endogenous variables into the agents' information sets.

The solution builds on Uhlig's (2001) undetermined coefficients method to solve full information DSGE models and the solutions of Nimark (2008), Melosi (2017) and Chapter 1 of this thesis for models with imperfect common knowledge. The advantage of using the undetermined coefficient methods is that it entails in a guess and verify strategy which is the standard procedure to solve models with imperfect common knowledge information (see Morris and Shin; 2002, for a discussion in a static setting).

Agents under imperfect common knowledge face the well-known complication that they have to form an infinite-regress of expectations of the expectations of others. This is true when agents extract information from a signal about an underlying fundamental from aggregate outcomes (see Townsend; 1983) or as a result of strategic interaction of agents with imperfect and dispersed information (see Morris and Shin; 2002). The solution follows the proposal of Nimark (2008) that truncates the order of higher-order expectations. As shown by Nimark (2017), this approximate solution is accurate for the class of models that importance of the order of expectation decreases with the order.

Despite of the substantially more complicated setting in expectation formation, one key implication of the solution method is that the endogenous persistence of state variables is the same under full information and imperfect common knowledge. This result implies that the inclusion of the endogenous state variables do not have to be included in fixed point solution for the reaction to higher-order expectations and their formation. As a result, the solution method is sufficiently fast to perform Bayesian estimation in a medium-scale DSGE model.

The proposed solution is an extension of other existing solution methods. By taking the variance of idiosyncratic signals to zero, the solution is a cumbersome way to compute the Uhlig's (2001) solution for full-information models. Regarding previous solution methods for ICK models, it extends the solution by including endogenous state variables in the system of equilibrium conditions. Moreover, in the case where agents observe only exogenous public and private signals, the solution is an extension of the framework of Blanchard et al. (2013). Their solution method is for imperfect information models with exogenous public signals. The proposed solution allows for imperfect and dispersed information.

The second contribution is empirical. As a primer exercise, the role of informational frictions relatively to standard frictions of full-information models for explaining business cycles is evaluated. Moreover, I study whether informational frictions are a alternative to replace the "reduced-form" frictions used in full information models to fit the data better.

For answering such questions, I develop a medium-scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007) with imperfect and dispersed information and compare with the full information benchmark. The study uses a comprehensive dataset regarding expectations as it includes expectations about output, consumption, investment and inflation.

The primary results of the empirical analysis are the following. The model under imperfect and dispersed information explain better the expectation data whereas the full information model explains better the macroeconomic aggregates. Second, informational frictions are more pronounced regarding the productivity and monetary shocks. However, the estimation suggests informational frictions are not substitutes of the standard real and nominal frictions. Instead, the combination of such frictions is important for explaining the expectation data.

Related literature

There are few attempts to evaluate the relevance of informational frictions quantitatively. Melosi (2014) estimates a model which firms have ICK about the money growth and technological shocks. The author shows that that the with ICK fits the data better than a model with sticky prices a la Calvo (1983) with indexation. Melosi (2017) estimates a sticky price model a la Calvo with ICK about monetary, technological, demand shocks. He finds that there is a signaling channel of monetary policy for the US and that the model fits better the data than a hybrid New Keynesian model with habits in consumption and indexation.

Both papers have the limitations that information frictions are assumed only for firms and that there is not endogenous persistence in the model. However, even without the latter, the models can describe better the data than the full information model with endogenous persistence (habits and/or indexation). The solution proposed here allows performing a more even comparison by introducing the same frictions for both models with full-information and ICK.

Reis (2009) and Mankiw and Reis (2010) estimate a sticky-information model with informational frictions not only firms but also households and workers. They find that the model with such informational frictions can mimic the dynamics of the main macroeconomic aggregates and show implications for monetary policy. Another interesting possibility is the method to solve DSGE models with rational inattention proposed by Maćkowiak and Wiederholt (2015). The calibrated model can match the empirical impulse response functions of an estimated VAR and full-information models.

One advantage of the approach of Reis (2009), Mankiw and Reis (2010) and Maćkowiak

and Wiederholt (2015) is that they allow households and firms to have different degrees of informational frictions. In the solution proposed here, all agents face the same information frictions. An extension for that case is left for future research.

Using a different approach for the solution, Angeletos et al. (forthcoming) propose a model with heterogeneous prior and using the limiting case of vanishing variance of private signals. In that way, they can solve and estimate the model by shutting down the learning process and representing the expectations hierarchy into a low-dimensional state variable. The advantage of this method is that it has the virtually the same computational burden of the full-information models. The drawbacks are that the dispersion in expectations is shut down and apply only to "confidence shocks" that not change fundamentals, but affects the economy though expectations.

The framework here is general. Agents can have noisy information about any of the shocks hitting the economy and can also incorporate "confidence shocks". Also, the proposed algorithm is more computationally expensive than its full-information counterpart, but fast enough to perform Bayesian estimation in a medium-scale DSGE model in a conventional notebook.

In the empirical part, I estimate a medium-scale DSGE model along the lines of Smets and Wouters (2007) under full information and under imperfect common knowledge. In the latter, both firms and households have imperfect and dispersed information about the shocks hitting the economy: total factor and investment-specific productivity, preference, government expenditure, wage and price mark-ups and monetary shocks.

For a first exploration of the information frictions, I use only exogenous private signals into agents' information sets. In that way, the computation is substantially faster than in the case of endogenous variables. Moreover, I use expectation data from the Survey of Professional forecasters to evaluate if informational frictions help not only describe standard macroeconomic aggregate but also expectations about them. With the notable exception of Melosi (2017), those papers do not use expectations data. While the latter only uses data on inflation expectations, this paper includes expectations about real GDP, consumption, investment and GDP deflator.

Given the findings of Coibion and Gorodnichenko (2012) described above, it is natural to include expectations data to evaluate if a model with noisy information can better explain the expectation formation than the standard full-information counterpart.

3.2 A general linear rational expectation model with imperfect common knowledge

There is a continuum of agents, $i \in [0,1]$ with imperfect common knowledge about the underlying unobserved shocks, i.e., the drivers of economic fluctuations as any standard DSGE model. In other words, there is a vector of shocks, x_t, which are not perfectly observed by those agents. They observe public and idiosyncratic signals about those shocks. Based on their imperfect and dispersed information, they form rational expectations about the underlying shocks and endogenous variables.

The equilibrium conditions of those agents are given by the following system of linear rational equations:

$$FE_t^{(1)}[Y_{t+1}] + G_1 Y_t + G_2 E_t^{(1)}[Y_t] + HY_{t-1} + LE_t^{(1)}[x_{t+1}] + M_1 x_t + M_2 E_t^{(1)}[x_t] = 0_{m\times 1} \quad (3.1)$$

where Y_t is a $m \times 1$ vector of endogenous variables and x_t is a $n \times 1$ vector of unobservable exogenous shocks. $E_t^{(1)}[\cdot] = \int_0^1 E_{it}[\cdot]di$ is the average expectation and $E_{it}[\cdot] = E[\cdot|\mathcal{I}_{it}]$ is agent i's expectation based on his information set at period t, $\mathcal{I}_{it}$.

This system enlarges the system of equations considered in previous chapters of this thesis by including the term HY_{t-1}. Unlike those chapters and the previous literature, it is allowed endogenous state variables in the system of equations.

Moreover, the dynamics of the shocks hitting the economy is given by

$$x_t = A_1 x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \Sigma_\varepsilon) \quad (3.2)$$

where Σ_ε is a $n \times n$ diagonal matrix. Throughout this paper, I assume that the matrix A_1 has all its eigenvalues lower than one, i.e., x_t is a stationary process.

The public signals are given by

$$s_t = c_{x,1} x_t + c_y Y_t + d_u u_t, \quad u_t \sim \mathcal{N}(0, \Sigma_u), \quad (3.3)$$

where Σ_u is a $p_u \times p_u$ diagonal matrix and s_t is a $p_u \times 1$ vector. This setup allows to include into agents' information sets, endogenous variables, and exogenous noisy public signals. Moreover, the idiosyncratic signals are given by

$$s_{it} = c_{x,2} x_t + d_v v_{it}, \quad v_{it} \sim \mathcal{N}(0, \Sigma_v) \quad (3.4)$$

where Σ_v is a $p_v \times p_v$ diagonal matrix and s_{it} is a $p_v \times 1$ vector.

The private signal introduces strategic uncertainty to agents. As discussed in previous chapters of this thesis, this implies that agents do not observe the endogenous variables, Y_t, since they are aggregate outcomes or relates to them due to equilibrium conditions. When c_y is not null, one can think s_t as containing some selected endogenous variables that are assumed to be observed by agents. Moreover, I assume that at the beginning of period t, Y_{t-1} is revealed. This assumption facilitates the inclusion of endogenous state variables since agents do not have to form expectations about the history of endogenous variables. Of course, it is also more realistic to assume that decision-makers can observe macroeconomic aggregates with a delay of one period.[1]

Formally, the information set of agent i at period t is given by

$$\mathcal{I}_{it} = \{s_\tau, s_{i\tau}, Y_{\tau-1} | \tau \leq t\}. \tag{3.5}$$

It is useful to define the k-th order expectation $E_t^{(k)}[\cdot]$ given information of period t about a variable x_t

$$E_s^{(k)}[x_t] = \int_0^1 E_{is}\left[E_s^{(k-1)}[x_t]\right] di, \tag{3.6}$$

for $k \geq 1$ and all s, t, with the convention $E_t^{(0)}[x_t] = x_t$.

Following Nimark (2008), denote the expectations hierarchy $x_t^{(0:\bar{k})} \equiv \left[\begin{array}{cccc} x_t' & E_t^{(1)}[x_t]' & \cdots & E_t^{(\bar{k})}[x_t]' \end{array}\right]'$ the vector that stacks the average higher-order expectations from order 0 to $\bar{k}$. $\bar{k}$ is last the order of average expectation that has impact on the equilibrium dynamics.

3.2.1 Equilibrium response to higher-order expectations

As standard in DSGE models, the solution is a recursive equilibrium law of motion given by

$$Y_t = PY_{t-1} + Qx_t^{(0:\bar{k})} \tag{3.7}$$

where P and Q are matrices to be determined by equilibrium conditions. Equation (3.7) is a natural extension for the full information case, which the endogenous variables respond only to the endogenous state variables and to shocks. Here, endogenous variables also respond to higher-order expectations of those shocks. Moreover, it extends the solutions of models

[1] If one think that it is unreasonable to assume such delay for perfectly observing a particular (or all) endogenous variables, one can easily include them in the public, s_t, without a noise.

with imperfect common knowledge such as fist two chapters of this thesis, Melosi (2017) and Nimark (2017) by including the recursive term.

Strategic uncertainty implies that higher-order expectations matter for the equilibrium outcomes under two circumstances. First, when agents extract information about an underlying fundamental from aggregate outcomes, i.e., c_y is not a null matrix (see Townsend; 1983, for a discussion). Second, when agents have strategic interaction, i.e., G_2 is not a null matrix (see Morris and Shin; 2002).

In turn, hierarchy of expectations is assumed to follow

$$x_t^{(0:\bar{k})} = Ax_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t \tag{3.8}$$

where A, B_ε and B_u are matrices to be determined by expectation formation conditions.

As discussed in previous chapters, expectations hierarchy follows a VAR(1) process that reaction both to structural shocks and the noise public signal. Individual expectations respond to both idiosyncratic and public signals' noises. Average expectation wash-out the response to idiosyncratic signals' noise whereas it preserves the response to public signal's noise. By the same reasoning, the response for latter is also preserved for higher-order expectations.

Intuitively, when forming expectations about average expectations, agents take into account that public signals also affect other agents' expectations. In the same manner, higher-order expectations of other agents' expectations also should take into account that everyone observes those shocks.

The following result relates the first-order expectation of $x_t^{(0:\bar{k})}$ with itself, which is useful to compute expectations about endogenous variables which are critical to the solution method.

Lemma 3.1. *The first-order expectation of the hierarchy of expectations satisfy*

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = Tx_t^{(0:\bar{k})} \tag{3.9}$$

where $T = \begin{bmatrix} 0_{n\bar{k}\times n} & I_{n\bar{k}} \\ 0_{n\times n} & 0_{n\times n\bar{k}} \end{bmatrix}$ *is the order transformation matrix.*

Proof. By the definition of $x_t^{(0:\bar{k})}$ one can see that $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = x_t^{(1:\bar{k}+1)}$. By definition for $\bar{k}$, any order s such that $s > \bar{k}$ does not affect the equilibrium. Then, without loss of generality, I can set $E_t^{(s)}[x_t] = 0$ if $s > \bar{k}$. Therefore, one can rewrite $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$ as

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \begin{bmatrix} x_t^{(1:\bar{k})} \\ E_t^{(\bar{k}+1)}[x_t] \end{bmatrix} = \begin{bmatrix} x_t^{(1:\bar{k})} \\ 0_{n\times 1} \end{bmatrix} = \begin{bmatrix} 0_{n\bar{k}\times n} & I_{n\bar{k}} \\ 0_{n\times n} & 0_{n\times n\bar{k}} \end{bmatrix} \begin{bmatrix} x_t \\ x_t^{(1:\bar{k})} \end{bmatrix} = Tx_t^{(0:\bar{k})}, \tag{3.10}$$

where in the first equality I used the definition of $x_t^{(1:\bar{k}+1)}$ and in the second equality I used that $E_t^{(\bar{k}+1)}[x_t] = 0$. In the last equality, T is defined accordingly.

□

Therefore, the first-order expectation of expectations hierarchy can be written as a linear transformation of the expectations hierarchy itself, given by the matrix T.

The following proposition shows the solution the equilibrium response of endogenous variables for a given guessed higher-order dynamics in equation (3.8).

Proposition 3.1. *For a given dynamics of higher-order expectations from equation (3.8), the system of equations (3.1) has a recursive equilibrium law of motion (3.7) such that:*

1. *P and Q satisfy the matrix equations:*

$$FP^2 + GP + H = 0, \tag{3.11}$$

$$FQAT + G_1Q + (FP + G_2)QT + (LA_1 + M_1)e_x'T + M_2e_x' = 0 \tag{3.12}$$

   where $G \equiv G_1 + G_2$ and $e_x \equiv \begin{bmatrix} I_n & 0'_{n\bar{k}\times n} \end{bmatrix}'$ is the selection matrix such that $x_t = e_x'x_t^{(0:\bar{k})}$.

2. *(Uhlig; 2001) P has a unique stable solution if all eigenvalues of P are smaller than unity in absolute value.*

3. *Given the solution of P, denote the matrix V such that:*

$$V = I_k \otimes G_1 + T' \otimes (FP + G_2) + T'A' \otimes F \tag{3.13}$$

   where $k = n \times (\bar{k}+1)$. Provided that there exists a inverse for V, the equilibrium solution for Q is given by

$$vec(Q) = V^{-1}vec\left((LA_1 + M_1)e_x'T + M_2e_x'\right), \tag{3.14}$$

   where $vec(\cdot)$ denotes columnwise vectorization.

Proof. By computing the individual expectation and integrating on i of the recursive law of motion (3.7), one can find that

$$\begin{aligned} E_t^{(1)}\left[Y_t\right] &= PE_t^{(1)}\left[Y_{t-1}\right] + QE_t^{(1)}\left[x_t^{(0:\bar{k})}\right] \\ &= PY_{t-1} + QTx_t^{(0:\bar{k})}. \end{aligned} \tag{3.15}$$

where last equality uses Lemma 3.1 and the fact that Y_{t-1} is observable at period t. Similarly, the average expectation for endogenous variables in $t+1$:

$$\begin{aligned} E_t^{(1)}\left[Y_{t+1}\right] &= PE_t^{(1)}\left[Y_t\right] + QE_t^{(1)}\left[x_{t+1}^{(0:\bar{k})}\right] \\ &= P^2Y_{t-1} + \left(PQT + QAT\right)x_t^{(0:\bar{k})} \end{aligned} \tag{3.16}$$

where last equality uses Lemma 3.1, equation (3.15) and the fact that $E_t^{(1)}\left[x_{t+1}^{(0:\bar{k})}\right] = AE_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$. Substituting the guessed solution (3.7) and the expectations (3.15-3.16) into the system of equations (3.1) one can find:

$$\left[FP^2 + GP + H\right]Y_{t-1} + \left[FQA + G_1Q + \left(FP + G_2\right)QT + FQAT + \left(LA_1e_x'T + M_1e_x'\right)\right]x_t^{(0:\bar{k})} = 0. \tag{3.17}$$

This condition must hold for all realizations of Y_{t-1} and $x_t^{(0:\bar{k})}$. Therefore, both coefficients between square brackets must be zero, which leads to conditions (3.11) and (3.12). □

This proposition connects the solution of imperfect common knowledge models with the standard undetermined coefficients solution for full information DSGE models. To see this, consider the case that all agents form expectations under perfect information. In that case, the average expectations is equal to the rational expectation under perfect information, $E_t^{(1)}[\cdot] = \mathbb{E}_t[\cdot]$. Therefore, the system of equations (3.1) collapse to

$$F\mathbb{E}_t\left[Y_{t+1}\right] + GY_t + HY_{t-1} + L\mathbb{E}_t\left[x_{t+1}\right] + Mx_t = 0_{m\times 1}, \tag{3.18}$$

where $M = M_1 + M_2$. Condition (3.11) is exactly the same as the usual "brute force" approach of Uhlig (2001)[2] to solve the endogenous persistence the linear rational expectations system (3.18). Therefore, P can be computed with standard techniques by turning condition (3.11) into a generalized eigenvalue and eigenvector problem (see Uhlig; 2001, for details). The key assumption for this result is that agents observe past endogenous variables at period t.

[2]The extension of the "sensitive" approach of Uhlig (2001) is not pursued in this paper. This method splits the variables into control and state variables, which is more efficient in terms of computing the persistence of endogenous variables. On the other hand, it is likely that this option is less efficient in computing the response to higher-order expectations. One would need to introduce an additional coefficient matrix (the response of state variables to the expectation hierarchy) to the fixed point iteration.

This result is stated in the following Corollary.

Proposition 3.1. *Consider the system of equations under imperfect common knowledge (3.1) and the corresponding system under full information (3.18). Then, the endogenous persistence of state variables, P, is the same under full information and imperfect common knowledge.*

Proof. Direct inspection of condition (3.11) and condition stated in Theorem 1 of Uhlig (2001). □

Corollary 3.1 has important implications for the solution method. The persistence of higher-order expectations affect only the response of endogenous response to those shocks, Q, but do not affect the endogenous persistence of state variables, P.

As it will be clear in the next section, the solution method will consist in a fixed point solution of equilibrium responses of the hierarchy expectation to shocks and public signals (A, B_ε and B_u) and the equilibrium responses of endogenous variables to those expectations (Q). This result implies that P can be computed only once outside of the fixed point iteration of those objects. This result dramatically speed-up the solution computation. Therefore, the fixed point algorithm developed in Chapter 1 can be adapted to solve the equilibrium response to higher-order expectations, Q.

Note that this result does not imply that higher-order expectations do not affect the persistence of endogenous variables. It only means that it does not affect how endogenous variables respond to state variables. The persistence of higher-order expectations still affects the persistence those endogenous variables via their response to the higher-order expectations.

Moreover, the formulation to solve Q is more computationally efficiency in comparison previous solution methods, as it leads to a linear relationship between the reaction of endogenous variables to higher-order average expectations (Q) and their persistence (A) that can be solved by straightforward vectorization.

Until now, the solution for Q is for a given guessed persistence of higher-order expectations, A. Next section, I characterize the expectation formation and show under which conditions A is a function of Q. In that case, there is a feedback between the optimal reaction to unobservable shocks and beliefs about them.

3.2.2 Expectations formation

Each agent i has to form expectations about the whole expectation hierarchy, $x_t^{(0:\bar{k})}$. Expectations are conditional on their information set (3.5). For now, I assume that past endogenous variables, $\{Y_{\tau-1}, \tau \leq t\}$, do not provide additional information about the $x_t^{(0:\bar{k})}$. In next section, this assumption is relaxed.

In that case, the observational equation relates the public and idiosyncratic signals to the expectations hierarchy such that

$$\begin{bmatrix} s_t \\ s_{it} \end{bmatrix} = \begin{bmatrix} c_{x,1}e_x' + c_y Q \\ c_{x,2}e_x' \end{bmatrix} x_t^{(0:\bar{k})} + \begin{bmatrix} c_y P \\ 0_{p_v \times m} \end{bmatrix} Y_{t-1} + \begin{bmatrix} d_u & 0_{p_u \times p_v} \\ 0_{p_v \times p_u} & d_v \end{bmatrix} \begin{bmatrix} u_t \\ v_{it} \end{bmatrix}. \tag{3.19}$$

where it was used the fact that $x_t = e_x' x_t^{(0:\bar{k})}$. This system can be rewritten simply as

$$S_{it} = C x_t^{(0:\bar{k})} + N Y_{t-1} + D_v v_{it} + D_u u_t, \tag{3.20}$$

where $S_{it} = \begin{bmatrix} s_t' & s_{it}' \end{bmatrix}'$ and C, N, D_v and D_u are defined accordingly. S_{it} is the vector of observed variables that provide information about the unobserved shocks and their higher-order expectations.

If C_y is not a null matrix, there are observed endogenous variables. In that case, the matrix C depends on the equilibrium response to higher-order expectations, Q. Therefore, the signal extraction problem has to deal with the endogenous response to shocks and their higher-order expectations.

One key assumption to simplify the expectation formation is that the structural shocks ε_t and noises u_t and v_{it} are normally distributed. This assumption and the fact that the system of equilibrium conditions is linear guarantees that rational expectations can be computed by the Kalman filter.

Each agent i applies the Kalman filter to form expectations $E_{it}[x_t^{(0:\bar{k})}]$ by using the guessed equation (3.8) as state equation and equation (3.20) as observational equation. Given the individual expectations it is possible to compute the average (first order) expectation about $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$. The following Lemma shows how those expectations are computed by the Kalman filter.

Lemma 3.2. *Each agent i using the observational equation given by (3.20) and guessed law of motion of $x_t^{(0:\bar{k})}$ given by (3.8) has the following individual expectation:*

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right) A E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon \varepsilon_t + \bar{K}\left(CB_u + D_u\right) u_t + \bar{K}D_v v_{it}. \tag{3.21}$$

where $\bar{K}$ is the steady-state Kalman gain is found by the solution of the Riccati equation resulting from

$$\bar{K} = \left[\bar{P}C' + \Sigma_{cov}\right]\left[C\bar{P}C' + \Sigma_{obs}\right]^{-1}, \tag{3.22}$$

and

$$\bar{P} = A\left[\bar{P} - \bar{K}\left(\bar{P}C' + \Sigma_{cov}\right)'\right]A' + \Sigma_{state}, \tag{3.23}$$

where $\Sigma_{obs} = D_v\Sigma_v D_v' + D_u\Sigma_u D_u'$, $\Sigma_{state} = B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u'$ *and* $\Sigma_{cov} = B_u\Sigma_u D_u'$.

Suppose that there is common knowledge of rationality. Then, average first order expectation of $x_t^{(0:\bar{k})}$ *is given by*

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{K}C\right)AE_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}\left(CB_u + D_u\right)u_t. \tag{3.24}$$

The proof is an application of the steady-state Kalman filter, which is left in the Appendix 3.A.1.

Lemma 3.2 shows how individual and average expectations about the expectations hierarchy are computed. Each agent i is rational and uses the Kalman filter to form expectations about the hierarchy of expectations. In order to compute the average expectations, it is essential to assume common knowledge of rationality. In other words, agents acknowledge that other agents are rational, and other agents acknowledge that other agents are rational and so on (for a formal definition is this context, see Nimark (2008)). Therefore, each agent i realizes that everyone else also form expectations in the same way and only differ by the realization of the noise v_{it} as shown in equation (3.21). In that way, average expectation about the expectation hierarchy from equation (3.24) is computing by taking the average of the continuum of agents, i.e., $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \int_0^1 E_{it}\left[x_t^{(0:\bar{k})}\right]di$ and the hierarchy expectations is consistent with equation (3.6).

In the same way as previous chapters, agents have to use the Kalman filter that takes into account the correlation of errors in the state equation and the observational equation (note that u_t appears in both equations). This is why the term Σ_{cov} enters in the equations (3.22) and (3.23).

In the following, the first-order expectation from equation (3.24) is used to verify the guessed dynamics for the hierarchy of expectations. The following Lemma will establishes a simple relation between the higher-order expectations vector with its first-order expectation and the underlying shock, which helps the mapping between the expectation in (3.24) and the guessed form (3.8).

Lemma 3.3. $x_t^{(0:\bar{k})}$ *can be rewritten as a linear function of* x_t *and its average expectation such that:*

$$x_t^{(0:\bar{k})} \equiv e_x x_t + T' E_t^{(1)} \left[x_t^{(0:\bar{k})} \right], \tag{3.25}$$

Proof. By the definition of $x_t^{(0:\bar{k})}$, one can be decomposed as $x_t^{(0:\bar{k})} = \begin{bmatrix} x_t' & (x_t^{(\bar{1}:\bar{k})})' \end{bmatrix}'$. By the same reason, the its average expectation can be decomposed as $E_t^{(1)} \left[x_t^{(0:\bar{k})} \right] = \begin{bmatrix} (x_t^{(\bar{1}:\bar{k})})' & E_t^{(\bar{k}+1)} [x_t]' \end{bmatrix}'$. Therefore, one can use this two equations to decompose $x_t^{(0:\bar{k})}$ such that

$$x_t^{(0:\bar{k})} \equiv \begin{bmatrix} x_t \\ x_t^{(\bar{1}:\bar{k})} \end{bmatrix} = \begin{bmatrix} I_n \\ 0_{n\bar{k}\times n} \end{bmatrix} x_t + \begin{bmatrix} 0_{n\times n\bar{k}} & 0_{n\times n} \\ I_{n\bar{k}} & 0_{n\bar{k}\times n} \end{bmatrix} \begin{bmatrix} x_t^{(1:\bar{k})} \\ E_t^{(\bar{k}+1)} [x_t] \end{bmatrix} = e_x x_t + T' E_t^{(1)} \left[x_t^{(0:\bar{k})} \right], \tag{3.26}$$

where last equality uses the definitions of T in Lemma 3.1 and e_x in Proposition 3.3. □

Lemma 3.3 states that the expectations hierarchy can be rewritten as a linear transformation of its expectation and the unobserved shocks. The following proposition shows that given the dynamics of $E_t^{(1)} \left[x_t^{(0:\bar{k})} \right]$ from Lemma 3.2 and the relation between $x_t^{(0:\bar{k})}$ and its own first-order expectation established in Lemma 3.3, it is possible to verify that $x_t^{(0:\bar{k})}$ follows the guessed structure in equation (3.8) and equilibrium conditions for A, B_ε and B_u.

Proposition 3.2. *Suppose that x_t is a stationary process and the higher-order expectations dynamics follow equation (3.8). Then, for a given matrix $C = C(Q)$ defined by equations (3.19-3.20), the equilibrium A, B_ε and B_u satisfy the matrix equations:*

$$\begin{aligned} \left(I_k - T' \bar{K} C \right) A &= e_x A_1 e_x' + T' \left(I_k - \bar{K} C \right) A T \\ \left(I_k - T' \bar{K} C \right) B_\varepsilon &= e_x \\ \left(I_k - T' \bar{K} C \right) B_u &= T' \bar{K} D_u \end{aligned} \tag{3.27}$$

such that $\bar{K}$ and $\bar{P}$ solve the Riccati equation resulting from equations

$$\begin{aligned} \bar{K} &= \left[\bar{P} C' + \Sigma_{cov} \right] \left[C \bar{P} C' + \Sigma_{obs} + C \Sigma_{cov} + \Sigma_{cov}' C' \right]^{-1} \\ \bar{P} &= A \left[\bar{P} - \bar{K} \left(\bar{P} C' + \Sigma_{cov} \right)' \right] A' + \Sigma_{state} \end{aligned} \tag{3.28}$$

where $\Sigma_{obs} = D_v \Sigma_v D_v' + D_u \Sigma_u D_u'$, $\Sigma_{state} = B_\varepsilon \Sigma_\varepsilon B_\varepsilon' + B_u \Sigma_u B_u'$ and $\Sigma_{cov} = B_u \Sigma_u D_u'$.

Proof. Substituting the average expectation from Lemma 3.2 into the expression from Lemma 3.3 one can find that:

$$x_t^{(0:\bar{k})} = e_x x_t + T' \left[\left(I_k - \bar{K}C \right) A E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] + \bar{K}CAx_t^{(0:\bar{k})} + \bar{K}CB_\varepsilon \varepsilon_t + \bar{K}(CB_u + D_u)u_t \right] \tag{3.29}$$

Then, using equation (3.2) and the fact that $x_t = e_x' x_t^{(0:\bar{k})}$ one can rewrite equation above as

$$\begin{aligned} x_t^{(0:\bar{k})} = e_x \left(A_1 e_x' x_{t-1}^{(0:\bar{k})} + \varepsilon_t \right) + T' \left(I_k - \bar{K}C \right) A E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] \\ + T'\bar{K}CAx_{t-1}^{(0:\bar{k})} + T'\bar{K}CB_\varepsilon \varepsilon_t + T'\bar{K}(CB_u + D_u)u_t \end{aligned} \tag{3.30}$$

Using Lemma 3.1 at period $t-1$ into equation above and rearranging:

$$x_t^{(0:\bar{k})} = \left[e_x A_1 e_x' + T' \left(I_k - \bar{K}C \right) AT + T'\bar{K}CA \right] x_{t-1}^{(0:\bar{k})} + \left[T'\bar{K}CB_\varepsilon + e_x \right] \varepsilon_t + \left[a\bar{K}(CB_u + D_u) \right] u_t. \tag{3.31}$$

This expression shows that the expectations hierarchy is a function of its lag, the structural shocks and the noise public signal, as guessed in equation (3.8). Therefore, the expression above verify that $x_t^{(0:\bar{k})}$ follows the guessed form and the square brackets terms provide identities for A, B_ε and B_u in equations (3.27) □

Proposition 3.2 states the conditions that need to be satisfied for the expectation formation be formed by common knowledge of rationality.

3.2.3 Equilibrium and solution algorithm

Proposition 3.1 shows how to compute the equilibrium response of endogenous variables to higher-order expectations and observable shocks for a given guessed dynamics for higher-order expectations. Moreover, for a given equilibrium response of endogenous variables, Proposition 3.2 shows how to compute the dynamics of higher-order expectations of agents with common knowledge of rationality.

The following Proposition states implied fixed point solution by the equilibrium conditions of the endogenous response to higher-order expectations and their formation.

Proposition 3.3. *The system of equations (3.1) whose agents form rational expectations based on the observation equations (3.20) has a recursive equilibrium law of motion (3.7) and a hierarchy of expectations*

$$
\begin{aligned}
Y_t &= PY_{t-1} + Qx_t^{(0:\bar{k})} \\
x_t^{(0:\bar{k})} &= Ax_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t,
\end{aligned}
\tag{3.32}
$$

such that P solves equation (3.11) from Proposition 3.1 and $(A, B_\varepsilon, B_u, Q)$ is the fixed point solution such that:

1. *If public signals are exogenous, i.e., $c_y = 0_{p_u \times m}$, then:*

   $(A, B_\varepsilon, B_u, \bar{K}, \bar{P})$ solves the fixed point from Proposition 3.2. Given the equilibrium A, the equilibrium Q solves equation (3.12) from Proposition 3.1

2. *If public signals are endogenous variables, i.e., $c_y \neq 0_{p_u \times m}$, then:*

   $(A, B_\varepsilon, B_u, \bar{K}, \bar{P}, Q)$ solves the fixed point including all equations from Proposition 3.2 and equation (3.12) from Proposition 3.1.

Proposition 3.3 states that P can be computed beforehand and its inclusion in the fixed point iteration is not required. This a direct implication of discussed in Corollary 3.1. Moreover, it discusses how is the fixed point solution of the DSGE model with imperfect common knowledge depends on the informational structure.

The case where public signals are exogenous ($c_y = 0_{p_u \times m}$), the matrix C in equation do not depend on Q. In words, the higher-order expectations can be computed independently agents' optimal decisions. The dynamics of the underlying shocks and the structure of exogenous signals if sufficient to form expectations Intuitively, shocks and the information about them is exogenous and, thus, do not depend on agents actions.

In the case of only exogenous signals, the solution is related to the solution strategy Blanchard et al. (2013) to solve imperfect information models with exogenous public signals. This particular case is an extension of their solution that allows both public and idiosyncratic exogenous signals.

In the more general case of endogenous public signals, the information is extracted from variables that aggregates (or related to variables that aggregate) individual agents' choices. In that case, the equilibrium response Q does not only depends on the persistence of higher-order expectations as shown in equation (3.12) but also affects how informative public signals are via the matrix C defined by equations (3.19-3.20). Therefore, there is a feedback between endogenous response and expectations, which implies that the equilibrium response Q has to be included in the fixed point iteration.

The fixed point stated in Proposition 3.3 can be solved by the fixed point iteration in the following algorithm.

Algorithm. *Set the initial values* $(A^{(0)}, B_{\varepsilon}^{(0)}, B_u^{(0)})$*, a small tolerance* $\epsilon > 0$ *and set* $i = 1$*. Compute* P *using equation (3.11). Then, follow the steps:*

1. *given* $A = A^{(i-1)}$ *and* P*, solve for* Q *the equation (3.12) by vectorization. Set* $Q^{(i)} = Q$*.*
2. *Given* $Q^{(i)}$*, construct the matrix* C *from the observation equation defined by (3.19-3.20). Set* $C^{(i)} = C$*.*
3. *Given* $B_{\varepsilon}^{(i-1)}$*,* $B_u^{(i-1)}$ *and* $C^{(i)}$ *compute* $\bar{K}$ *and* $\bar{P}$ *using equations (3.28). Set* $\bar{K}^{(i)} = \bar{K}$ *and* $\bar{P}^{(i)} = \bar{P}$*.*
4. *Given* $B_{\varepsilon}^{(i-1)}$*,* $B_u^{(i-1)}$*,* $A^{(i-1)}$*,* $C^{(i)}$ *and* $K^{(i)}$*, solve for* B_{ε}*,* B_u*,* A *the equations (3.27) by matrix inversion. Set* $B_{\varepsilon}^{(i)} = B_{\varepsilon}$*,* $B_u^{(i)} = B_u$*,* $A^{(i)} = A$*.*
5. *if min* $\|B_{\varepsilon}^{(i)} - B_{\varepsilon}^{(i-1)}\| < \epsilon$*,* $\|B_u^{(i)} - B_u^{(i-1)}\| < \epsilon$*,* $\|A^{(i)} - A^{(i-1)}\| < \epsilon$ *and* $\|P^{(i)} - P^{(i-1)}\| < \epsilon$*, stop iterating or else set* $i = i + 1$ *and go back to step 1.*

If the public signals are exogenous, $c_y = 0_{p_u \times m}$*, then the algorithm can be simplified by skipping steps 1 and 2 and computing* Q *only after finding the fixed point to* $(A, B_{\varepsilon}, B_u, \bar{K}, \bar{P}, Q)$*.*

Discussion

The solution method relies in the existence of a $\bar{k}$, such that the expectations of orders $s > \bar{k}$ do not play a role in the equilibrium outcomes. Nimark (2017) shows that $\bar{k}$ exists for a class of rational expectation model as long as the importance of order decreases with the order. In Section 3.3.1, I discuss that the medium-scale DSGE model used in the application is included in the class of models that the order has decreasing relevance for the equilibrium outcomes.

The solution method that he proposes do not allow for endogenous variables in the system of equations. One might consider if his results apply for this solution method. It turns out that Corollary 3.1 implies that objects in the fixed point solution are the same as his solution method. Therefore, there is no reason why his result should not apply to the solution proposed here.

The proposed solution method improves the computational efficiency within the fixed point iteration. Melosi (2017) writes the system of linear rational expectation in terms of the higher-order expectations by substituting the guessed solution into the equilibrium conditions. This implies that the matrix coefficients of such system (the counterpart of

matrices F, G_1, G_2, L, M_1 and M_2) depend on the guessed response Q. As a result, it leads to a non-linear relationship between Q and A.

One implication of the solution is that such additional complexity is not required. It is possible to solve the model under imperfect common knowledge using only the first-order expectation in the system of equations as in (3.1). As shown in 3.2, this method closely related to standard undermined coefficients method to solve full-information DSGE models. Therefore, for each iteration, Q can be computed by straightforward vectorization.

The algorithm to find the fixed point solution is similar to the one proposed by Melosi (2017). One key difference is that Lemma 3.2 allows for a much simpler characterization of the fixed point condition of the expectation formation as stated in Proposition 3.2, and previously in Chapter 1. Such characterization implies that, for each iteration, A, B_ε and B_u can be computed by simple matrix inversion. This increases the computational efficiency in comparison with Melosi (2017) strategy which implies in solving non-linear equations at each iteration.

3.2.4 Learning from past endogenous variables

Until now, it was assumed that past endogenous variables do not provide additional information about the unobserved exogenous variables and their higher-order expectations. Most of the literature on imperfect information and imperfect common knowledge has not considered this possibility. One exception is Nimark (2008) which firms to choose their optimal prices, extract information from past output and inflation to form expectations about future inflation and aggregate marginal cost.

There are two reasons why past endogenous variables can provide additional information. First, due to the endogenous persistence, they can provide information about current endogenous variables that are unobserved under imperfect common knowledge. That channel is already embedded in conditions of Proposition 3.1.

Second, past endogenous variables provide information about the past expectations hierarchy. Since there is persistence in higher-order expectations, those variables can be informative about the hierarchy of expectations. In section 3.2.2, this possibility was not considered. This section, relax this assumption.

Suppose that there is a subset of the endogenous variables that observing their past provide information about $x_t^{(0:\bar{k})}$. Denote those variables as $Y_{t-1}^{obs} = c_l Y_{t-1}$, where c_l is $p_l \times m$ a selection matrix. The case where agents observe all past endogenous variables is when $c_l = I_m$

Using the guessed solution (3.7) in $t-1$, the observational equations now given by

$$\begin{bmatrix} s_t \\ Y_{t-1}^{obs} \\ s_{it} \end{bmatrix} = \begin{bmatrix} c_{x,1}e_x' + c_y Q \\ 0_{p_l \times k} \\ c_{x,2}e_x' \end{bmatrix} x_t^{(0:\bar{k})} + \begin{bmatrix} 0_{p_v \times k} \\ c_l Q \\ 0_{p_u \times k} \end{bmatrix} x_{t-1}^{(0:\bar{k})} + \begin{bmatrix} c_y P & 0_{p_u \times m} \\ 0_{p_l \times m} & c_l P \\ 0_{p_v \times m} & 0_{p_v \times m} \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \end{bmatrix} + \begin{bmatrix} d_u & 0_{p_u \times p_v} \\ 0_{p_l \times p_u} & 0_{p_l \times p_v} \\ 0_{p_v \times p_u} & d_v \end{bmatrix} \begin{bmatrix} u_t \\ v_{it} \end{bmatrix}, \tag{3.33}$$

In that case, the observation equation depends also on $x_{t-1}^{(0:\bar{k})}$. There are two alternatives to the optimal filtering in this case. The first and more standard approach is to redefine the state variable as $\hat{x}_t \equiv \begin{bmatrix} (x_t^{(0:\bar{k})})' & (x_{t-1}^{(0:\bar{k})})' \end{bmatrix}'$. This would double the size of the state space, which is computationally costly for a medium-scale model. The second alternative is to use Nimark's (2015) version of Kalman filter that allows lagged state variables in the observational equation. The second approach is followed because it is more computationally efficient and can be adapted in the solution method proposed here. The equation above can be simply rewritten as

$$Z_{it} = C_1 x_t^{(0:\bar{k})} + C_2 x_{t-1}^{(0:\bar{k})} + \hat{N} w_{t-1} + \hat{D}_v v_{it} + \hat{D}_u u_t, \tag{3.34}$$

where $w_{t-1} = \begin{bmatrix} Y_{t-1}' & Y_{t-2}' \end{bmatrix}'$ is a vector of predetermined variables.

Again, each agent i uses the Kalman filter to form expectations $E_{it}[x_t^{(0:\bar{k})}]$. Given the individual expectations it is possible to compute the average (first order) expectation about $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$. The following Lemma shows how those expectations are computed by the Kalman filter.

Lemma 3.4. *Each agent i with the observational equation given by (3.20) and the guessed law of motion of $x_t^{(0:\bar{k})}$ given by (3.8) has the following individual expectation:*

$$\begin{aligned} E_{it}\left[x_t^{(0:\bar{k})}\right] = &\left[\left(I_k - \hat{K}C_1\right)A - \hat{K}C_2\right] E_{it-1}\left[x_t^{(0:\bar{k})}\right] \\ &+ \hat{K}\left[(C_1 A + C_2)x_{t-1}^{(0:\bar{k})} + C_1 B_\varepsilon \varepsilon_t + \hat{D}_v v_{it} + (C_1 B_u + \hat{D}_u)u_t\right] \end{aligned} \tag{3.35}$$

where $\hat{K}$ is the steady-state Kalman gain is found by the solution of the Riccati equation resulting from

$$\begin{aligned} \hat{K} = &\left[A\hat{P}(C_1 A + C_2)' + \Sigma_{state} C_1' + \Sigma_{cov}\right] \\ &\left[(C_1 A + C_2)\hat{P}(C_1 A + C_2)' + C_1 \Sigma_{state} C_1' + \Sigma_{obs} + C_1 \Sigma_{cov} + \Sigma_{cov}' C_1'\right]^{-1} \end{aligned} \tag{3.36}$$

and

$$\hat{P} = A\hat{P}A' + \Sigma_{state} - \hat{K}\left[A\hat{P}(C_1A + C_2)' + \Sigma_{state}C_1' + \Sigma_{cov}\right]' \tag{3.37}$$

where $\Sigma_{obs} = \hat{D}_v\Sigma_v\hat{D}_v' + \hat{D}_u\Sigma_u\hat{D}_u'$, $\Sigma_{state} = B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u'$ *and* $\Sigma_{cov} = B_u\Sigma_u\hat{D}_u'$.

Suppose that there is common knowledge of rationality. Then, average expectation of $x_t^{(0:\bar{k})}$ *is given by*

$$\begin{aligned} E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = &\left[\left(I_k - \hat{K}C_1\right)A - \hat{K}C_2\right]E_{it-1}\left[x_t^{(0:\bar{k})}\right] \\ &+ \hat{K}\left[(C_1A + C_2)x_{t-1}^{(0:\bar{k})} + C_1B_\varepsilon\varepsilon_t + (C_1B_u + \hat{D}_u)u_t\right] \end{aligned} \tag{3.38}$$

The proof is a application of the steady-state Kalman filter shown in Nimark (2015), which is left in the Appendix 3.A.2.

Lemma 3.4 shows how individual and average expectations about the expectations hierarchy are computed when agents observe lagged endogenous variables. If $c_l = 0_{p_l \times k}$, one can see in equations (3.33-3.34) that $C_2 = 0_{p_v+p_u+p_l \times k}$. In that case, the expectation formation from Lemma 3.4 is exactly the same as the one in Lemma 3.2.[3]

The following proposition shows the counterpart of Proposition 3.2 when agents observe lagged endogenous variables.

Proposition 3.4. *Suppose that* x_t *is a stationary process and the higher-order expectations dynamics follow equation (3.8). Then, for a given matrices* $C_1 = C_1(Q)$ *and* $C_2 = C_2(Q)$ *defined by equations (3.33-3.34), the equilibrium* A, B_ε *and* B_u *satisfy the matrix equations:*

$$\begin{aligned} \left(I_k - T'\hat{K}C_1\right)A &= e_xA_1e_x' + T'\hat{K}C_2 - T'\hat{K}C_2T + T'\left(I_k - \hat{K}C_1\right)AT \\ \left(I_k - T'\hat{K}C_1\right)B_\varepsilon &= e_x \\ \left(I_k - T'\hat{K}C_1\right)B_u &= T'\hat{K}\tilde{D}_u \end{aligned} \tag{3.39}$$

such that $\hat{K}$ *and* $\hat{P}$ *solve the Riccati equation resulting from equations*

$$\begin{aligned} \hat{K} = &\left[A\hat{P}(C_1A + C_2)' + \Sigma_{state}C_1' + \Sigma_{cov}\right] \\ &\left[(C_1A + C_2)\hat{P}(C_1A + C_2)' + C_1\Sigma_{state}C_1' + \Sigma_{obs} + C_1\Sigma_{cov} + \Sigma_{cov}'C_1'\right]^{-1} \\ \hat{P} = &A\hat{P}A' + \Sigma_{state} - \hat{K}\left[A\hat{P}(C_1A + C_2)' + \Sigma_{state}C_1' + \Sigma_{cov}\right]' \end{aligned} \tag{3.40}$$

[3]Rearranging the terms of $\hat{K}$ and $\hat{P}$ using the definitions of $P_{t|t}$ and $P_{t|t-1}$ from Appendix 3.A.2, one can see that $\hat{K}$ and $\hat{P}$ becomes the same as $\bar{K}$ and $\bar{P}$ from Lemma 3.2.

where $\Sigma_{obs} = D_v \Sigma_v D_v' + D_u \Sigma_u D_u'$, $\Sigma_{state} = B_\varepsilon \Sigma_\varepsilon B_\varepsilon' + B_u \Sigma_u B_u'$ *and* $\Sigma_{cov} = B_u \Sigma_u D_u'$

Proof. Substituting the average expectation from Lemma 3.4 into the expression from Lemma 3.3 one can find that:

$$\begin{aligned} x_t^{(0:\bar{k})} = e_x x_t + T' \left[\left(I_k - \hat{K} C_1 \right) A - \hat{K} C_2 \right] E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] \\ + T' \hat{K} \left[C_1 A + C_2 \right] x_{t-1}^{(0:\bar{k})} + T' \hat{K} C_1 B_\varepsilon \varepsilon_t + \hat{K} (C_1 B_u + D_u) u_t \end{aligned} \tag{3.41}$$

Then, using equation (3.2) and the fact that $x_t = e_x' x_t^{(0:\bar{k})}$ one can rewrite equation above as

$$\begin{aligned} x_t^{(0:\bar{k})} \equiv e_x' \left(A_1 e_x x_{t-1}^{(0:\bar{k})} + \varepsilon_t \right) + T' \left[\left(I_k - \hat{K} C_1 \right) A - \hat{K} C_2 \right] E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] \\ + T' \hat{K} \left[C_1 A + C_2 \right] x_{t-1}^{(0:\bar{k})} + T' \hat{K} C_1 B_\varepsilon \varepsilon_t + T' \hat{K} (C_1 B_u + D_u) u_t. \end{aligned} \tag{3.42}$$

Using Lemma 3.3 at period $t-1$ into equation above and rearranging:

$$\begin{aligned} x_t^{(0:\bar{k})} = \left[e_x' A_1 e_x + T' \left(I_k - \hat{K} C \right) A T - T' \hat{K} C_2 T + T' \hat{K} (C_1 A + C_2) \right] x_{t-1}^{(0:\bar{k})} \\ + \left[T' \hat{K} C_1 B_\varepsilon + e_x' \right] \varepsilon_t + \left[T' \hat{K} (C_1 B_u + D_u) \right] u_t. \end{aligned} \tag{3.43}$$

This expression shows that the expectations hierarchy is a function of its lag, the structural shocks and the noise public signal, as guessed in equation (3.8). Therefore, the expression above verify that $x_t^{(0:\bar{k})}$ follows the guessed form and the square brackets terms provide identities for A, B_ε and B_u in equations (3.39) □

Lemma 3.4 shows how individual and average expectations about the expectations hierarchy are computed when agents observe lagged endogenous variables. If $c_l = 0_{p_l \times k}$, one can see in equations (3.33-3.34) that $C_2 = 0_{p_v + p_u + p_l \times k}$. In that case, the expectation formation from Lemma 3.4 is exactly the same as the one in Lemma 3.2.[4]

The additional information from past endogenous variables affects the equilibrium in two ways. First, it affects the Kalman gain $\hat{K}$, i.e., how agents optimally update their expectation based on observed variables. Second, it changes the persistence of higher-order expectations as there are two additional terms in the equation (3.39) for A.

The equilibrium in this case, would be given by the Proposition 3.3 but instead of using Proposition 3.2 to compute the higher-order expectations, now one need to compute using Proposition 3.4.

[4]Rearranging the terms of $\hat{K}$ and $\hat{P}$ using the definitions of $P_{t|t}$ and $P_{t|t-1}$ from Appendix 3.A.2, one can see that $\hat{K}$ and $\hat{P}$ becomes the same as $\bar{K}$ and $\bar{P}$ from Lemma 3.2.

3.3 A primer evaluation of informational frictions

The literature on informational frictions is quite rich regarding the potential drivers of business cycles. As a first exploration of the information frictions under imperfect and dispersed information, I introduce exogenous private signals into agents' information sets into an otherwise standard medium-scale DSGE model. There are no shocks other than the standard shocks from the full information model. Specifically, there are no public signals (as in Lorenzoni (2009), Blanchard et al. (2013)) or common shocks in private information that generates fluctuations in expectations that are not linked with structural shocks (as in Angeletos and La'O (2013) and Angeletos et al. (forthcoming)).

It worthy consider whether the additional informational friction can improve the performance of the model regarding better explaining the data without relying on further shocks. In the following, I evaluate whether informational frictions can help to describe standard macroeconomic aggregate and expectations data.

3.3.1 Model

The model is a standard medium-scale DSGE model along the lines of Christiano et al. (2005), Smets and Wouters (2007). The model features sticky prices and wages with partial indexation to past inflation, habit formation in consumption, investment adjustment costs and variable capital utilization.

Seven exogenous structural shocks drive the dynamics of the economy. It includes total factor and investment-specific productivity shocks on the technological side and preference and government expenditure on the demand side. There are also wage and price mark-up shocks, and a monetary policy shock.

I deviate from the literature by assuming that agents have imperfect and dispersed information. Specifically, households and firms do not perfectly observe the structural shocks. Instead, they receive noisy idiosyncratic signals about such shocks.

Timing

Time is discrete and each period contains two stages. In the first stage, shocks and signals realize and intermediate goods firms choose their optimal price based on information from the signals. Also, given the information from their signals, households choose their consumption, investment, installed capital and its utilization level, and set their wage for their differentiated labor supply. In the second stage, rental rates and wages of differentiated labor are revealed. Competitive final good firms use intermediate goods to sell the final good to households.

Competitive labor packers use the supply of differentiated labor from households and sell a homogeneous labor package for intermediate firms. The latter, choose their demand for capital and labor to produce the intermediate good.

This timing ensures two features. First, the market clearing in all markets. Competitive final good firms ensure that the supply of final good adjusts to consumption and investment demand. Given prices of intermediate goods, differentiated wages and rental rate chosen at stage 1, firms allocate capital and labor to accommodate the demand from final good firms. Finally, given the wages set at stage 1, labor packers aggregate demand the differentiated labor services from households and supply homogeneous labor to intermediate firms such that they can clear the labor market.

Second, intermediate good firms do not use information from their production side to extract information from aggregate variables. Therefore, both intermediate firms and households use only information from their signals to form expectations.

Final good firms

The homogeneous final good, Y_t, is a composite of intermediate goods, Y_{it}, indexed by $i \in [0,1]$, such that

$$Y_t = \left(\int_0^1 (Y_{it})^{\frac{1}{1+\mu_t^p}} \, di \right)^{1+\mu_t^p}, \tag{3.44}$$

where μ_t^p is the time-varying price mark-up of intermediate goods such that $log(1+\mu_t^p) = log(1+\mu^p) + \eta_t^p$. In turn, and η_p^p follows the process

$$\eta_t^p = \rho_p \eta_{t-1}^p + \varepsilon_t^p, \quad \varepsilon_t^p \sim \mathcal{N}(0, \sigma_p^2). \tag{3.45}$$

Final goods producers are perfectly competitive firms that sell their goods to households for a price P_t. Therefore, they maximize profits

$$P_t Y_t - \int_0^1 P_{it} Y_{it} di \tag{3.46}$$

subject to (3.44). Thus, the demand for each intermediate firm i is given by

$$Y_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\frac{1+\mu_t^p}{\mu_t^p}} Y_t, \tag{3.47}$$

where $P_t = \left(\int_0^1 (P_{it})^{-\frac{1}{\mu_t^p}} \, di \right)^{-\mu_t^p}$ is the final good price level.

Intermediate good firms

Each intermediate producers is a monopolistic competitive firm $i \in [0,1]$ who uses labor and capital to produce their goods. Their production function is given by

$$Y_{it} = e^{a_t}(K_{i,t})^{\alpha}(\gamma^t L_{it})^{1-\alpha} - \gamma^t \Phi_p, \tag{3.48}$$

where γ represents the labor-augmenting deterministic growth rate of productivity and a_t is a productivity shock. K_{it} and L_{it} denote the amount of capital and labor demanded by firm i at period t and Φ_p is a fixed cost included in the production function.[5] The productivity shock follows

$$z_t = \rho_z z_{t-1} + \varepsilon_t^z, \quad \varepsilon_t^z \sim \mathcal{N}(0, \sigma_z^2). \tag{3.49}$$

Following Calvo (1983), each firm i is subject to price stickiness such that each period firms can choose their price optimally with a probability $1 - \xi_p$. Firms not allowed to do so, use a partial indexation rule given by

$$P_{i,t} = (\Pi_{t-1})^{\iota_p} \left(\bar{\Pi}\right)^{1-\iota_p} P_{i,t-1}, \tag{3.50}$$

where $\Pi_t = P_t/P_{t-1}$ is the gross inflation and $\bar{\Pi}$ is its steady-state value. Firms able to choose prices, P_{it}^*, maximize their expected discounted flow of profits, which implies the following optimization problem:

$$E_{it} \sum_{s=0}^{\infty} (\beta \xi_p)^s \Lambda_{t,t+s} \left[\left(X_{t,t+s} P_{it}^* - MC_{i,t+s} \right) Y_{i,t+s} \right], \tag{3.51}$$

subject to $Y_{i,t+s} = \left(\frac{X_{t,t+s} P_{it}^*}{P_{t+s}}\right)^{-\frac{1+\mu_{t+s}^p}{\mu_{t+s}^p}} Y_{t+s}$. $\Lambda_{t,t+s}$ denotes household's stochastic discount factor between periods t and $t+s$ and $X_{t,t+s}$ is the indexation between the same periods. $MC_{i,t}$ denotes the marginal cost of firm i at period t. Consistently with the indexation rule (3.50), $X_{t,t+s}$ is given by

$$X_{t,t+s} = \begin{cases} \bar{\Pi}^{(1-\iota_p)s} \prod_{j=1}^{s} \left(\Pi_{t+j-1}^{\iota_p}\right) & if \quad s \geq 1 \\ 1 & if \quad s = 0. \end{cases} \tag{3.52}$$

Given the optimal prices and the indexation rule, the price level has the following law of motion

[5] The fixed cost must be scaled by γ^t. Otherwise, the fixed cost would become increasingly smaller over time in relative terms since the production grows at a constant rate γ in the steady-state.

$$P_t = \left[\xi_p \left(\Pi_{t-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} P_{t-1} \right)^{-\frac{1}{\mu_t^p}} + (1-\xi_p) \left(\int_0^1 P_{it}^* di \right)^{-\frac{1}{\mu_t^p}} \right]^{-\mu_t^p}. \tag{3.53}$$

Labor packers

There is a continuum of households indexed by $h \in [0,1]$ who supplies differentiated labor services that are imperfect substitute for other households' labor services. As in Erceg et al. (2000), it is assumed that there are "employment agencies" that combines households' labor supply using the following aggregator

$$L_t = \left(\int_0^1 L_{ht}^{\frac{1}{1+\mu_t^w}} dh \right)^{1+\mu_t^w}, \tag{3.54}$$

where μ_t^w denote the agency's wage mark-up such that $log(1+\mu_t^w) = log(1+\mu^w) + \eta_t^w$. In turn, η_t^w follows the process

$$\eta_t^w = \rho_w \eta_{t-1}^w + \varepsilon_t^w, \quad \varepsilon_t^w \sim \mathcal{N}(0, \sigma_w^2). \tag{3.55}$$

Agencies pay for each household h their wage, W_{ht}, and sell a homogeneous labor service to intermediate firms at a cost, W_t. Therefore, they maximize profits

$$W_t L_t - \int_0^1 W_{it} L_{it} di \tag{3.56}$$

subject to (3.54). Thus, the demand labor for each household h is given by

$$L_{ht} = \left(\frac{W_{ht}}{W_t} \right)^{-\frac{1+\mu_t^w}{\mu_t^w}} L_t, \tag{3.57}$$

where $W_t = \left(\int_0^1 (W_{it})^{-\frac{1}{\mu_t^w}} di \right)^{-\mu_t^w}$ is the nominal wage index.

Households

Each household $h \in [0,1]$ derives utility from leisure and consumption. In order to maximize their expected utility, they choose consumption ($C_{h,t}$), labor ($L_{h,t}$), government bonds ($B_{h,t}$) as well the installed capital level ($K_{h,t}$) and its utilization rate, ($U_{h,t}$). The capital rented to firms, K_t^u, is determined by the installed capital and the utilization ratio.

The preferences of each household, h, is given by the utility function

$$U = E_{ht}\left[\sum_{s=0}^{\infty}\beta^s e^{\eta^c_{t+s}}\left(ln\left(C_{h,t+s}-\varphi C_{t+s-1}\right)-\frac{L^{1+\sigma_L}_{h,t+s}}{1+\sigma_L}\right)\right], \tag{3.58}$$

where $C_{h,t}$ is the consumption and $L_{h,t}$ denotes the supply of differentiated labor services of household h at period t. Households have external habit preferences which is determined by parameter φ and σ_L is the inverse Frisch elasticity. $E_{ht}[\cdot]$ is the household h expectation operator and η^c_t is a preference shock given by

$$\eta^c_t = \rho_c\eta^c_{t-1}+\varepsilon^c_t, \quad \varepsilon^c_t \sim \mathcal{N}(0,\sigma^2_c). \tag{3.59}$$

The capital stock, $K_{h,t}$ is owned by household h following the law of motion

$$K_{h,t} = (1-\delta)K_{h,t-1}+e^{\eta^I_t}\left(1-S(I_{h,t}/I_{h,t-1})\right)I_{h,t}, \tag{3.60}$$

where $S(I_t/I_{t-1})$ is adjustment investment cost function that determines the share of investment which does not become new capital. Following Christiano et al. (2005), the cost function $S(\cdot)$ is a function of the growth of investment with the properties, $S(\gamma) = S'(\gamma) = 0$, and $S''(\cdot) = s''$. η^I_t is a stationary investment-specific technological shock given by

$$\eta^i_t = \rho_i\eta^i_{t-1}+\varepsilon^i_t, \quad \varepsilon^i_t \sim \mathcal{N}(0,\sigma^2_i) \tag{3.61}$$

Households rent capital to firms an effective amount of capital, $K^u_{h,t}$, given by

$$K^u_{h,t} = U_{h,t}K_{h,t-1}, \tag{3.62}$$

where $U_{h,t}$ is degree of capital utilization whose cost in terms of the final good is given by function $a(U_{h,t})K_{h,t-1}$. As in Christiano et al. (2005), this function has the properties $a(\bar{U}) = a(\bar{U})' = 0$ and $a(\bar{U})'' = a''$, where $\bar{U}$ is the utilization value at steady-state. In compensation, households receive a return R^k_t on the effective rented capital.

Therefore, the household is subject to the budget constraint

$$P_tC_{h,t}+P_tI_{h,t}+B_{h,t}+P_ta(u_{h,t})\bar{K}_{h,t}+Q_{t+1,t}A_{h,t} \leq R_{t-1}B_{h,t-1}W_{h,t}L_{h,t}+R^k_tu_{h,t}\bar{K}_{h,t}+A_{h,t-1}+T_{h,t}, \tag{3.63}$$

in each period. $T_{h,t}$ denote net transfers, $A_{h,t}$ denotes a vector of one-period state-contingent securities and $Q_{t+1,t}$ is the price of such asset.

Each household provides a differentiated labor service for labor agency. Therefore, households choose their wage by setting a mark-up over the marginal rate of substitution between consumption and labor. As in Erceg et al. (2000) there is a probability ξ_w that households do not optimize their wages. Households that are not able to optimize, update their wage using a partial indexation rule given by

$$W_{h,t} = (\Pi_{t-1})^{\iota_w} \left(\bar{\Pi}\right)^{1-\iota_w} \gamma W_{h,t-1}, \tag{3.64}$$

which relates real wages to the productivity growth and past inflation.

Therefore, each household minimizes their expected discounted labor disutility such that

$$E_{ht}\left[\sum_{s=0}^{\infty}(\beta\xi_w)^s \left(-\frac{L_{j,t+s}^{1+\sigma_L}}{1+\sigma_L}\right)\right] \tag{3.65}$$

subject to the budget constraint (3.63) at all periods $s \in [0, \infty)$ and to the period $t+s$ nominal wage $W_{h,t+s} = X_{t,t+s}^{w} W_{h,t}^{*}$ and

$$X_{t,t+s}^{w} = \begin{cases} (\gamma\bar{\Pi}^{1-\iota_w})^s \prod_{j=1}^{s} \left(\Pi_{t+j-1}^{\iota_w}\right) & if \quad s \geq 1 \\ 1 & if \quad s = 0. \end{cases} \tag{3.66}$$

Given the optimal prices and the indexation rule, the price level has the following law of motion

$$W_t = \left[\xi_w \left(\gamma\Pi_{t-1}^{\iota_w}\bar{\Pi}^{1-\iota_w} W_{t-1}\right)^{-\frac{1}{\mu_t^w}} + (1-\xi_w)\left(\int_0^1 W_{ht}^{*} dh\right)^{-\frac{1}{\mu_t^w}}\right]^{-\mu_t^w}. \tag{3.67}$$

The assumption of a complete set of state-contingent securities guarantees that all households h will make the same consumption and saving choices. This is true despite the fact that they have differentiated wages and form expectations based on idiosyncratic signals. This assumption is for tractability. In that way, it is not required to keep track of the stationary distribution of capital as in heterogeneous agents model. Thus, I can apply standard techniques of log-linearization to solve DSGE models and still have informational frictions in the households' decisions.

Government Policies

The central bank follows a nominal interest rate rule by adjusting the interest rate using a simple Taylor rule with partial adjustment according to

$$\frac{R_t}{\bar{R}} = \left(\frac{R_{t-1}}{\bar{R}}\right)^{\phi_R} \left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\right]^{(1-\phi_R)} e^{\eta_t^r} \tag{3.68}$$

where variables with a bar denote the steady-state values and η_t^r is the monetary policy shock, which follows

$$\eta_t^r = \rho_r \eta_{t-1}^r + \varepsilon_t^r, \quad \varepsilon_t^r \sim \mathcal{N}(0, \sigma_r^2). \tag{3.69}$$

For simplicity, this specification assumes that central bank responds to output deviation to the steady-state value instead of the potential output.

The government budget constraint follows

$$P_t G_t + R_{t-1} B_{t-1} = B_t + T_t \tag{3.70}$$

where T_t and B_t aggregates across all households the lump-sum taxes and bonds, respectively. Government expenditure follows a simple stochastic process such that $G/Y = g_y + \eta_t^g$, where g_y is the steady-state share of government expenditure to output and η_t^g is a expenditure shock given by

$$\eta_t^g = \rho_g \eta_{t-1}^g + \varepsilon_t^g, \quad \varepsilon_t^g \sim \mathcal{N}(0, \sigma_g^2). \tag{3.71}$$

Resource constraint and market clearing

The aggregate resource constraint

$$C_t + I_t + G_t + a(U_t) K_t = Y_t, \tag{3.72}$$

can be found by combining the zero profit condition of final goods firms, labor packers and the government budget constraint into households' budget constraint and integrating over h.

The market clearing in labor and capital markets ensures that

$$\begin{aligned} K_t^u &= \int_0^1 K_{it} di \\ L_t &= \int_0^1 L_{it} di. \end{aligned} \tag{3.73}$$

The bond supply and the transfers

$$\begin{aligned} B_t &= \int_0^1 B_{ht} dh \\ T_t &= \int_0^1 T_{ht} dh, \end{aligned} \tag{3.74}$$

are consistent with the government expenditure and the budget constraint (3.70).

Information

Intermediate good firms and households are subject to information frictions. Specifically, firms and household receive noisy idiosyncratic signal about the structural shocks such that

$$x_{j,t}^s = \eta_t^s + v_{j,t}, \quad v_{j,t} \sim \mathcal{N}(0, \sigma_{v,j}^2), \tag{3.75}$$

where $s \in \{a, c, i, g, p, w, r\}$ denote each type of shock and $j \in [0, 1]$ is a index that pools both intermediate good firms i and households h. This assumption implies that firms and household receive signals with the same properties and are subject to the same informational frictions. Each firm or household j's information set is given by

$$\mathcal{I}_t^j = \{x_{j,s}^a, x_{j,s}^c, x_{j,s}^i, x_{j,s}^g, x_{j,s}^p, x_{j,s}^w, x_{j,s}^r | s \leq t\}. \tag{3.76}$$

Since there is a continuum of households, $h \in [0, 1]$, and a continuum of firms $i \in [0, 1]$, the average expectation for firms and households is the same and denoted by $E_t^{(1)}[\cdot] = \int_0^1 E_{j,t}[\cdot] dj$, where $E_{j,t}[\cdot] = E[\cdot|\mathcal{I}_t^j]$.

For the empirical analysis, it is useful to define the noise-to-signal ratio, $r_j = \frac{\sigma_{v,j}}{\sigma_j}$ where $j \in \{a, c, i, g, p, w, r\}$. It is well-known that it is the noise-to-signal ratio that determines the learning process in signal extraction problems. In other words, for all combinations of $\sigma_{v,j}$ and σ_j that implies the same r_j, there is the same equilibrium response to a unit shock of η_t^j.

Optimality conditions

Households

The first order conditions for consumption and bonds are

$$\Lambda_{h,t}E_{ht}[P_t] = \frac{E_{ht}\left[e^{\eta_t^c}\right]}{C_{h,t} - \varphi C_{h,t-1}} \tag{3.77}$$

$$\Lambda_{h,t} = \beta E_{ht}\left[\Lambda_{h,t+1}R_t\right] \tag{3.78}$$

where $\Lambda_{h,t}$ is the Lagrange multiplier of the budget constraint (3.63). The optimal conditions for capital and investment are given by

$$\Phi_{h,t} = \beta E_{ht}\left[\Lambda_{h,t+1}\left(R_{t+1}^k U_{h,t+1} - P_{t+1}a(U_{h,t+1})\right)P_{t+1}\right] + (1-\delta)\beta E_{ht}\left[\Phi_{h,t+1}\right] \tag{3.79}$$

$$\begin{aligned}\Lambda_{h,t}E_{ht}[P_t] = \Phi_{h,t}E_{ht}\left[e^{\eta_t^i}\right]&\left(1 - S\left(\frac{I_{h,t}}{I_{h,t-1}}\right) - S'\left(\frac{I_{h,t}}{I_{h,t-1}}\right)\frac{I_{h,t}}{I_{h,t-1}}\right) \\ &+ \beta E_{ht}\left[\Phi_{h,t+1}e^{\eta_{t+1}^i}S'\left(\frac{I_{h,t+1}}{I_{h,t}}\right)\left(\frac{I_{h,t+1}}{I_{h,t}}\right)^2\right]\end{aligned} \tag{3.80}$$

where $\Phi_{h,t}$ is the Lagrange multiplier of the capital accumulation constraint. The first order condition for capital utilization is

$$E_{ht}[R_t^k] = a'(U_{h,t})E_{ht}[P_t] \tag{3.81}$$

Recall that since households can buy state-contingent assets in terms of consumption (but not for leisure). Therefore, it must hold that for all households $h \in [0,1]$ that $C_{h,t} = C_t$, $I_{h,t} = I_t$, $U_{h,t} = U_t$, $\Lambda_{h,t} = \Lambda_t$ and $\Phi_{h,t} = \Phi_t$. However, households still have heterogeneous expectations, $E_{ht}[\cdot]$, wages, $w_{h,t}$, and labor supply $L_{h,t}$.

The optimal condition for setting the wage of differentiated labor, $W_{h,t}^*$ is

$$E_{ht}\left[\sum_{s=0}^{\infty}(\beta\xi_w)^s\left(\Lambda_{t,t+s}\frac{X_{t,t+s}^w}{\mu_{t+s}^w} - \frac{1+\mu_{t+s}^w}{\mu_{t+s}^w}\frac{L_{h,t+s}^{\sigma_L}}{W_{h,t}^*}\right)L_{h,t+s}\right] = 0 \tag{3.82}$$

where $\Lambda_{t,t+s} = \frac{\Lambda_{t+s}}{\Lambda_t}$ is the stochastic discount factor from period t to period $t+s$. In the following, it will be useful to define the marginal rate of substitution of labor and consumption give by

$$MRS_{ht} = \frac{\left(C_t - \varphi C_{t-1}\right)^{-1}}{L_{h,t+s}^{\sigma_L}} \tag{3.83}$$

Intermediate good firms

At stage 1, firms choose their optimal price based on their information. The optimal condition for intermediate good firm i price is

$$E_{it}\left[\sum_{s=0}^{\infty}(\beta\xi_p)^s\Lambda_{t,t+s}\left(\frac{X_{t,t+s}}{\mu^p_{t+s}}-\frac{1+\mu^p_{t+s}}{\mu^p_{t+s}}\frac{MC_{i,t+s}}{P^*_{it}}\right)Y_{i,t+s}\right]=0. \tag{3.84}$$

At stage 2, they hire labor services from the labor packer at the nominal wage, W_t, and rent capital from households at the rental rate, R^k_t. Cost minimization subject to production function (3.48) implies that the capital-labor ratio must satisfy[6]

$$\frac{K_{it}}{L_{it}}=\frac{\alpha}{1-\alpha}\frac{W_t}{R^k_t}. \tag{3.85}$$

Therefore, each firm i marginal cost is given by

$$MC_{it}=\frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\frac{(R^k_t)^\alpha\,(W_t)^{1-\alpha}}{a_t}. \tag{3.86}$$

Note that the constant returns of scale production function implies that firms have the same capital-labor and marginal costs.

Detrending and log-linearized model

Log-linearization around steady-state requires variables to be stationary. However, there are two trends in the model. All real variables grow along with the productivity trend whereas nominal variables grow along with the price level. Therefore, stationarity is induced by detrending those variables.

For variables real variables such as Y_t, C_t, I_t, K_t, K^u_t, they are detrended using the following procedure

$$\hat{Z}_t=\frac{Z_t}{\gamma^t}, \tag{3.87}$$

for any real variable Z_t. One exception is the Lagrange multiplier of the budget constraint, Λ_t, that has to be scaled up, i.e., $\hat{\Lambda}_t=\Lambda_t\gamma^t$ (since consumption grows at rate γ).

Nominal variables such as $\hat{\Lambda}_t$, $\hat{W}_t$ and R^k_t, the detrending procedure implies that

$$\hat{Z}_t=\frac{Z_t}{P_t}, \tag{3.88}$$

[6]Note that there is no expectation operator for W_t and R^k_t as they are revealed at stage 2.

for any nominal variable Z_t.

Finally, for the variables that are already stationary such as Π_t, R_t, L_t and U_t as well the transformed variables above, I take the log deviation from steady-state

$$z_t = log(Z_t/\bar{Z}), \tag{3.89}$$

for any stationary variable Z_t. Therefore, the lower case variables denote log-deviation from steady-state of the upper case variables.

The log-linearization of conditions optimal conditions leads to the following system of equations

$$c_t = \left(\frac{h/\gamma}{1+h/\gamma}\right) c_{t-1} + \left(\frac{1}{1+h/\gamma}\right) E_t^{(1)}[c_{t+1}] - \left(\frac{1-h}{1+h/\gamma}\right) E_t^{(1)}[r_t - \pi_{t+1}] + E_t^{(1)}[\eta_{t+1}^c - \eta_t^c] \tag{3.90.1}$$

$$i_t = \left(\frac{1}{1+\beta}\right) i_{t-1} + \left(\frac{\beta}{1+\beta}\right) E_t^{(1)}[i_{t+1}] - \left(\frac{1}{s''(1+\beta)\gamma^2}\right) E_t^{(1)}[q_t + \eta_t^i] \tag{3.90.2}$$

$$q_t = \left(\frac{\beta(1-\delta)}{\gamma}\right) E_t^{(1)}[q_{t+1}] + \left(1 - \frac{\beta(1-\delta)}{\gamma}\right) E_t^{(1)}[r_{t+1}^k] - E_t^{(1)}[r_t - \pi_{t+1}] \tag{3.90.3}$$

$$k_t = \left(\frac{\beta(1-\delta)}{\gamma}\right) k_{t-1} + \left(1 - \frac{\beta(1-\delta)}{\gamma}\right) \left(i_t + s''(1+\beta)\gamma^2 \eta_t^i\right) \tag{3.90.4}$$

$$u_t = \left(R^k/a'\right) E_t^{(1)}[r_t^k] \tag{3.90.5}$$

$$k_t^u = k_{t-1} + u_t \tag{3.90.6}$$

$$mrs_t = \left(\frac{1}{(1-h/\gamma)}\right) (c_t - (h/\gamma)c_{t-1}) + \gamma_l l_t \tag{3.90.7}$$

$$y_t = \Phi_p(\alpha k_t^u + (1-\alpha) l_t + a_t) \tag{3.90.8}$$

$$k_t^u - l_t = w_t - r_t^k \tag{3.90.9}$$

$$mc_t = \alpha r_t^k + (1-\alpha) w_t - a_t \tag{3.90.10}$$

$$y_t = c_y c_t + i_y i_t + u_y u_t + g_t \tag{3.90.11}$$

$$r_t = \phi_r r_{t-1} + (1-\phi_r)\left(\phi_\pi \pi_t + \phi_y y_t\right) + \eta_t^r \tag{3.90.12}$$

$$\pi_t = \iota_p \xi_p \pi_{t-1} + \left(1 - \xi_p(1+\beta\iota_p)\right) E_t^{(1)}[\pi_t] + \kappa_p \xi_p E_t^{(1)}[mc_t + \mu_t^p] + \beta \xi_p E_t^{(1)}[\pi_{t+1}] \tag{3.90.13}$$

$$w_t + \pi_t = \xi_w \left(w_{t-1} + \iota_w \pi_{t-1}\right) + (1 - \xi_w(1+\beta)) E_t^{(1)}[w_t] + (1 - \xi_w(1+\beta\iota_w)) E_t^{(1)}[\pi_t] \tag{3.90.14}$$

$$- \kappa_w \xi_w E_t^{(1)}[w_t - mrs_t - \mu_t^w] + \beta \xi_w E_t^{(1)}[w_{t+1} + \pi_{t+1}]$$

where $c_y = C/Y$, $c_i = I/Y$, $\kappa_p = \frac{(1-\xi_p)(1-\beta\xi_p)}{\xi_p}$, $\kappa_w = \frac{(1-\xi_w)(1-\beta\xi_w)}{\xi_w(1+\gamma_l\theta_w)}$ and $\theta_w = \frac{\mu^w}{\mu^w-1}$. All conditions are well-known in macroeconomic models. The first equation is the Euler equation for consumption by combining the log-linearized versions of equations (3.77) and (3.78). The second equation is the equilibrium condition for investment from (3.80) and third is the evolution of the marginal real value of a unit of capital, q_t, that comes from condition (3.79). The forth is log-linearized version of the law of motion of installed capital (3.60). The fifth is the log-linearized version of the optimal condition for capital utilization (3.81) and the sixth comes from the definition of utilized capital (3.62). The seventh and eighth are log-linearized version of the aggregation of the marginal rate of substitution between consumption and labor from equation (3.83) and the production function (3.48), respectively. The ninth and tenth comes from aggregating firms' cost minimization condition (3.85) and marginal cost (3.85), respectively. The eleventh comes from the aggregate resource constraint (3.72) whereas twelfth comes from the Taylor rule (3.68). Finally, the thirteenth is the New Keynesian Phillips curve for prices that combines log-linearized versions of equations (3.84) and (3.53). Similarly, the last is the New Keynesian Phillips curve for wages that combines log-linearized versions of equations (3.82) and (3.67).

There are some differences from the full information case that it worthy comment. First, expectations in those conditions are the average expectation $E_t^{(1)}[\cdot] \int_0^1 E_{jt}[\cdot]dj$. Second, unlike the full information case, all equations include expectations of period t endogenous variables and shocks. For instance, consider the Euler equation. It includes expectation about the nominal interest rate ($E_t^{(1)}[r_t]$) and about the preference shock ($E_t^{(1)}[\eta_t^c]$). In the full-information counterpart, the Euler includes those variables without the expectation operator.

Third, the New Keynesian Phillips curves (NKPC) for prices and wages (equations 3.90.13 and 3.90.14), implies that higher-order average expectations matter for the equilibrium outcomes. To see this, by taking average expectations of equation (3.90.13) and substituting it iteratively, one can find that

$$\pi_t = \iota_p\xi_p\pi_{t-1} + \sum_{k=1}^{\infty}\left(1-\xi_p(1+\beta\iota_p)\right)^{k-1}\left[\kappa_p\xi_p E_t^{(k)}[mc_t+\mu_t^p] + \beta\xi_p E_t^{(k)}[\pi_{t+1}]\right] \tag{3.91}$$

When $\iota_p = 0$, there is no indexation for firms not allowed to optimize. In that case, equation (3.91) this is the NKPC augmented by higher-order expectations from Nimark (2008). Firms have to form not only expectations about marginal cost, price's mark-up shock, and future inflation, but also higher-order expectations about those objects.

Applying the same procedure to equation (3.90.14), one can find

$$w_t = \xi_w \left(w_{t-1} + \iota_w \pi_{t-1}\right) + \sum_{k=1}^{\infty} \left(1 - \xi_w(1+\beta)\right)^{k-1} \left[(1 - \xi_w(1+\beta\iota_w)) E_t^{(k)}[\pi_t] - E_t^{(k-1)}[\pi_t]\right] \tag{3.92}$$

$$\sum_{k=1}^{\infty} \left(1 - \xi_w(1+\beta)\right)^{k-1} \left[-\kappa_w \xi_w E_t^{(k)}[w_t - mrs_t - \mu_t^w] + \beta \xi_w E_t^{(k)}[w_{t+1} + \pi_{t+1}]\right]$$

This equation augments the NKPC for wages with higher-order expectations. In a similar way as firms, households must form higher-order expectations about marginal rate of substitution between consumption and labor, wage's mark-up shock, and future inflation and real wages.

Note that the weights in higher-order expectations decreases with the order for both equations (3.91) and (3.92).[7] Therefore, the solution method introduced in Section 3.2 can be applied to solve the model.

In such case, the expectation hierarchy, $x_t^{(0:\bar{k})}$, is given by

$$x_t^{(0:\bar{k})} = \left[\begin{array}{cccc} x_t' & E_t^{(1)}[x_t]' & E_t^{(2)}[x_t]' & \cdots & E_t^{(\bar{k})}[x_t]' \end{array} \right]' \tag{3.93}$$

where $x_t = [\eta_t^a \quad \eta_t^c \quad \eta_t^i \quad \eta_t^g \quad \eta_t^p \quad \eta_t^w \quad \eta_t^r]$. During the estimation procedure, I fix $\bar{k} = 6$. This number is sufficiently high such that the difference of including an additional order changes the impulse response functions in order of magnitude of 10^{-3}. Including additional orders is perfectly possible, but it increases substantially the computational burden of the estimation.[8]

Finally, equations (3.91) and (3.92) also highlights one of the advantages of the proposed solution method. The methods of Nimark (2008) and Melosi (2017) requires to use equations (3.91) and (3.92) instead of equations (3.90.13) and (3.90.14) into the system of equilibrium equations. Such system of equations would have all higher-order expectations the endogenous variables. With the proposed method from Section 3.2, one can use the system of equations (3.90), which includes only the first-order average expectation.

It is straightforward to see that the system of equilibrium conditions (3.90) can be written in matricial form as in equation (3.1). Therefore, the techniques developed in Section 3.2 can be applied to solve and then estimate the model.

[7] Since $\beta \in (0,1)$, $\xi_p \in (0,1]$ and $\iota_p \in [0,1]$, then $(1 - \xi_p(1+\beta\iota_p)) \in (-\beta, 1)$. Moreover, $\xi_w \in (0,1]$ implies that $(1 - \xi_w(1+\beta+\kappa_w)) \in (-\beta, 1)$.

[8] Each additional order increases the expectation hierarchy size in 7, which is the number of shocks.

3.3.2 Empirical analysis

The empirical analysis is the following. I estimate the model under full information and imperfect common knowledge using two datasets. The first dataset consists in only macroeconomic variables. The second dataset includes both macroeconomic and expectation data.

Following Del Negro and Eusepi (2011), by estimating the models with those two datasets, I can formally evaluate how well the models fit the macroeconomic data and expectation data separately, despite the fact that both datasets are used during the estimation. Specifically, the marginal likelihood with the macroeconomic variables only evaluates how much each model fits for that dataset. By computing the marginal likelihood for complete dataset and taking the difference with the latter, one can assess the fit of the model only for expectation data but still using information of both types of data.

Data

The data used in estimation includes macroeconomic aggregates such as GDP, consumption and investment, real wages, hours worked, inflation and nominal interest rate. The expectation data includes expectations about 1-quarter ahead of GDP, consumption, investment, and inflation. A detailed description of the data and transformation is left in Appendix 3.B. The data is quarterly from 1981:Q4-2017Q4.

The measurement equations that links data with the models variables is given by

$$d_{y,t}^{data} = \bar{\gamma} + \Delta y_t \tag{3.94.1}$$
$$d_{c,t}^{data} = \bar{\gamma} + \Delta c_t \tag{3.94.2}$$
$$d_{i,t}^{data} = \bar{\gamma} + \Delta i_t \tag{3.94.3}$$
$$d_{w,t}^{data} = \bar{\gamma} + \Delta w_t \tag{3.94.4}$$
$$L_t^{data} = l_t \tag{3.94.5}$$
$$\Pi_t^{data} = \bar{\pi} + \pi_t \tag{3.94.6}$$
$$R_t^{data} = \bar{r} + r_t \tag{3.94.7}$$
$$Error_{dy,q=1,t}^{data} = y_t - E_{t-1}^{(1)}[y_t] + \varepsilon_{y,t}^{me} \tag{3.94.8}$$
$$Error_{dc,q=1,t}^{data} = c_t - E_{t-1}^{(1)}[c_t] + \varepsilon_{c,t}^{me} \tag{3.94.9}$$
$$Error_{di,q=1,t}^{data} = i_t - E_{t-1}^{(1)}[i_t] + \varepsilon_{i,t}^{me} \tag{3.94.10}$$
$$Error_{\pi,q=1,t}^{data} = \pi_t - E_{t-1}^{(1)}[\pi_t] + \varepsilon_{\pi,t}^{me} \tag{3.94.11}$$

where the variables with superscript "data" counterparts of the variables in the database. $d_{j,t}^{data}$ denote the quarterly growth rate in the data for variables $j \in \{y, c, i, w\}$. $\bar{\gamma} = 100(\gamma - 1)$, $\bar{\pi} = 400(\Pi - 1)$ and $\bar{\pi} = 400(R - 1)$ are the steady-state values for net quarterly growth, and net annualized inflation and interest rate, respectively. The variables $Error_{j,q=1,t}^{data}$ for $j \in \{y, c, i, \pi\}$ are the 1-step ahead forecast error of such variables for the expectation formed at period $t - 1$.

Measurement errors for expectation data are introduced to avoid stochastic singularity. The measurement errors are denotes as σ_j^{me} for $j \in \{y, c, i, \pi\}$.[9]

When using the dataset with macroeconomic variables only, the measurement equations consists of the first seven equations from (3.94). When using the dataset including expectations, all equations (3.94) are included as measurement equations.

Calibration and Priors

Following Smets and Wouters (2007), parameters that would be difficult to estimate are kept fixed during the estimation. The depreciation rate δ is set to 0.025 on a quarterly basis. The government expenditure share, g_y, is calibrated the historical mean value of 18%. Also, the steady-state mark-up in the labor market is set such that $\mu^w = 1.5$. Finally, consistently with national accounts, the capital share of the production function, α, is set to 0.3.

Since it is likely that the introduction of informational frictions affects the estimation of the real and nominal frictions, I do not rely much on previous estimates to form priors. Therefore, most of the priors are agnostic and not much informative.

Table (3.1) shows the prior distribution for each parameter. It shows the choice of mean and standard deviation as well the implied mode and 90% interval of the distribution.

Parameters that are between 0 and 1 such as the shocks' persistence, partial adjustment of the Taylor rule, the probability of readjusting prices and wages, indexation for prices and wages, and habit parameters have a prior with distribution beta with a mean of 0.5 and standard deviation of 0.2. In other words, loose priors centered in the middle point (see the 90% confidence bands on Table (3.1)).

The trend growth, inflation rate, and nominal interest rate steady-state parameters have a normally distributed priors with mean equals to the historical average. The standard deviations are 0.20 for the former and 1.00 for the others, which leads to loose priors.

The priors on the adjustment cost of investment, capital utilization and the inverse of Frisch elasticity are taken from Del Negro et al. (2007). The parameters for the Taylor rule

[9]By including the four forecast errors, the dataset has 11 variables but the model has 7 shocks. This leads to stochastic singularity that prevents the estimation of the model. The inclusion of measurement errors is a standard procedure in this case.

Table 3.1: Prior distributions

Par	Type	Mean	Std. Dev.	Mode	90% Interval		Par	Type	Mean	Std. Dev.	Mode	90% Interval	
Common parameters							**Common parameters**						
Endogenous propagation and steady-state							Exogenous shocks parameters						
s''	$\mathcal{G}$	4.00	1.50	3.44	1.89	6.74	ρ_a	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
γ_l	$\mathcal{G}$	2.00	0.75	1.72	0.95	3.37	ρ_i	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
φ	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83	ρ_c	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
ξ_p	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83	ρ_g	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
ξ_w	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83	ρ_p	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
ι_w	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83	ρ_w	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
ι_p	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83	ρ_R	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83
a''	$\mathcal{G}$	0.20	0.10	0.15	0.07	0.39	σ_a	$\mathcal{B}$	0.05	2.00	0.00	0.02	0.12
ϕ_p	$\mathcal{G}$	1.25	0.12	1.24	1.06	1.45	σ_c	$\mathcal{IG}$	0.05	2.00	0.00	0.02	0.12
ϕ_π	$\mathcal{G}$	1.50	0.25	1.46	1.11	1.93	σ_i	$\mathcal{IG}$	1.00	2.00	0.64	0.35	2.42
ϕ_r	$\mathcal{B}$	0.50	0.20	0.50	0.17	0.83	σ_g	$\mathcal{IG}$	0.05	2.00	0.00	0.02	0.12
ϕ_y	$\mathcal{G}$	0.20	0.10	0.15	0.07	0.39	σ_p	$\mathcal{IG}$	0.05	2.00	0.00	0.02	0.12
$\bar{\pi}$	$\mathcal{G}$	2.35	1.00	1.92	0.98	4.20	σ_w	$\mathcal{IG}$	0.25	2.00	0.03	0.08	0.62
$\bar{r}$	$\mathcal{G}$	4.26	1.00	4.03	2.76	6.03	σ_R	$\mathcal{IG}$	0.15	2.00	0.01	0.05	0.37
$\bar{\gamma}$	$\mathcal{G}$	0.40	0.20	0.30	0.14	0.78							
Parameters for ICK model only							**Measurement errors**						
Noise-to-signal ratio of private signals							σ_y^{me}	$\mathcal{IG}$	0.02	0.10	0.00	0.01	0.05
r_a	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36	σ_c^{me}	$\mathcal{IG}$	0.02	0.10	0.00	0.01	0.05
r_c	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36	σ_i^{me}	$\mathcal{IG}$	0.10	0.10	0.01	0.04	0.22
r_i	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36	σ_π^{me}	$\mathcal{IG}$	0.10	0.10	0.01	0.04	0.22
r_g	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36							
r_p	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36							
r_w	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36							
r_R	$\mathcal{G}$	1.00	0.70	0.51	0.18	2.36							

Note: $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{IG}$ denote the Gamma, Beta and Inverse Gamma distribution.

have priors with the gamma distribution with are standard values of $\phi_\pi = 1.5$ and $\phi_\pi = 0.2$ for the mean and values of 0.25 and 0.1, respectively, for the standard deviations.

Priors for standard deviations are distributed as inverse gamma. Following the procedure of Del Negro et al. (2007), the means are chosen to match the second moments of the observed macroeconomic variables in a pre-sample from 1960:Q1 to 1980Q4. The standard deviations for the priors are equal to 2 to ensure loose priors around the mean.

For the imperfect common knowledge model, it is also required to choose priors for the noise-to-signal ratio for each shock. The priors for such ratios are distributed as gamma with mean 1 and standard deviation 0.7. Noise-to-signal ratios of 1 imply that agents take roughly one year to learn the actual values of the shocks hitting the economy. The shape of the distribution ensures a large mass to values close to zero, where the ICK model is equivalent to the full information model (see that the median and the 5% percentile are roughly 0.5 and 0.18, respectively).

Finally, measurement errors priors are distributed as inverse gamma with mean to match roughly 5% of the total variance of each expectation data. The standard deviations are small to ensure tight priors to avoid the measurement errors to explain most of the variation of the expectation data.

Posterior estimates

Standard Bayesian estimation techniques that combine the prior distributions assumed in Table 3.1 with the likelihood function computed using the Kalman filter to form the posterior distribution. As standard in the literature, posterior distribution is computed using the MCMC methods[10]. Specifically, I use the random block Random-Walk Metropolis-Hastings (block-RWMH) algorithm to draw from the posterior density.

Table 3.2 shows the posterior estimates under both types of model and for both datasets. For each model and dataset, it shows the posterior mean and the 90% highest posterior density (HPD) interval.

The estimation of the full information model in the dataset with only macroeconomic variables is consistent with the results from Smets and Wouters (2007) and Del Negro et al. (2007). However, there are still some important differences. The parameter φ is relatively high compared with the standard value of 0.7 in the literature. Such value is found using tight priors around 0.6-0.7 while our prior is loose enough to allow higher values for that parameter. Given the loose priors for price and wage rigidity parameters, the estimates are relatively higher than the literature. However, results are consistent the agnostic priors from

[10]MCMC denotes Markov-Chain Monte Carlo.

Table 3.2: Posterior estimates

	Dataset: Macroeconomic only						Dataset: Macroeconomic and Expectations					
	Full information			ICK model			Full information			ICK model		
	Mean	90% Interval		Mean	90% Interval		Mean	90% Interval		Mean	90% Interval	
Endogenous propagation and steady-state parameters												
s''	4.64	2.95	6.66	3.82	2.42	5.35	6.76	5.22	8.49	7.01	5.84	8.31
γ_l	1.91	1.38	2.56	2.21	1.36	3.25	3.02	2.40	3.74	5.21	4.12	6.55
φ	0.93	0.89	0.97	0.95	0.91	0.98	0.98	0.98	0.99	0.97	0.96	0.98
ξ_p	0.86	0.83	0.88	0.98	0.97	0.99	0.75	0.71	0.79	0.96	0.95	0.96
ξ_w	0.70	0.62	0.76	0.76	0.62	0.88	0.71	0.67	0.75	0.66	0.61	0.71
ι_w	0.62	0.31	0.88	0.36	0.12	0.63	0.18	0.04	0.39	0.51	0.18	0.82
ι_p	0.10	0.03	0.21	0.11	0.02	0.30	0.08	0.02	0.16	0.05	0.01	0.10
a''	0.48	0.30	0.71	0.81	0.43	1.33	0.36	0.22	0.56	0.21	0.13	0.33
ϕ_p	1.34	1.20	1.48	1.50	1.34	1.67	1.22	1.10	1.35	1.44	1.29	1.58
ϕ_π	2.16	1.80	2.56	1.45	1.13	1.79	2.39	2.10	2.72	2.10	1.92	2.31
ϕ_r	0.85	0.80	0.89	0.88	0.82	0.93	0.73	0.66	0.78	0.68	0.62	0.73
ϕ_y	0.07	0.03	0.11	0.27	0.16	0.41	0.12	0.08	0.16	0.05	0.04	0.06
$\bar{\pi}$	2.20	1.05	3.43	1.99	1.17	2.64	1.56	0.68	2.69	1.64	0.91	2.46
$\bar{r}$	6.30	4.95	7.79	6.68	5.51	7.90	6.04	4.82	7.39	4.32	3.13	5.60
$\bar{\gamma}$	0.36	0.27	0.45	0.52	0.48	0.55	0.47	0.41	0.52	0.42	0.40	0.43
Exogenous shocks parameters												
ρ_a	0.91	0.86	0.96	0.94	0.89	0.98	0.996	0.991	0.999	0.95	0.93	0.96
ρ_i	0.86	0.76	0.94	0.79	0.64	0.90	0.20	0.15	0.25	0.12	0.08	0.16
ρ_c	0.25	0.10	0.42	0.19	0.12	0.27	0.18	0.13	0.23	0.15	0.10	0.19
ρ_g	0.988	0.976	0.996	0.993	0.986	0.998	0.98	0.97	0.98	0.997	0.995	0.999
ρ_p	0.97	0.96	0.99	0.70	0.44	0.89	0.995	0.991	0.998	0.63	0.57	0.69
ρ_w	0.27	0.10	0.45	0.43	0.27	0.63	0.15	0.05	0.26	0.48	0.42	0.54
ρ_R	0.51	0.36	0.66	0.55	0.45	0.64	0.56	0.41	0.71	0.64	0.57	0.71
σ_a	0.42	0.37	0.48	0.42	0.37	0.47	0.53	0.47	0.60	0.51	0.45	0.57
σ_c	0.26	0.23	0.30	0.55	0.45	0.66	0.32	0.28	0.36	0.42	0.34	0.53
σ_i	0.75	0.61	0.92	1.21	0.97	1.51	1.47	1.29	1.66	1.64	1.42	1.90
σ_g	0.34	0.30	0.39	0.33	0.29	0.38	0.35	0.31	0.39	0.38	0.33	0.43
σ_p	0.05	0.04	0.06	0.08	0.03	0.15	0.12	0.09	0.16	0.11	0.09	0.14
σ_w	0.35	0.28	0.41	0.38	0.24	0.58	0.37	0.31	0.43	0.32	0.26	0.40
σ_R	0.14	0.12	0.16	0.12	0.11	0.14	0.17	0.14	0.20	0.17	0.15	0.20
Noise-to-signal ratio of private signals												
r_a				0.80	0.15	1.87				2.19	1.82	2.56
r_c				0.50	0.35	0.66				0.15	0.03	0.30
r_i				0.06	0.01	0.15				0.04	0.01	0.10
r_g				0.86	0.63	1.09				0.47	0.34	0.61
r_p				0.18	0.03	0.46				0.30	0.12	0.48
r_w				0.22	0.03	0.59				0.09	0.02	0.21
r_R				1.84	0.69	3.40				0.59	0.23	0.98
Measurement errors												
σ_y^{me}							0.06	0.05	0.08	0.05	0.04	0.06
σ_c^{me}							0.03	0.02	0.04	0.03	0.02	0.03
σ_i^{me}							1.15	0.88	1.48	1.24	0.97	1.54
σ_π^{me}							0.42	0.32	0.55	0.42	0.32	0.55

Note: Estimates are based on 400.000 draws of the block-RWMH algorithm using three blocks. Half of the draws are discarded as burn-in. The variance from the candidate distribution is computed by the inverse of the Hessian of the log posterior at its minimum value.

Del Negro and Schorfheide (2008). The authors show that nominal rigidities are not robust for different specifications of priors.

When comparing the full information estimates for both datasets (columns 1 and 3 from Table 3.2), it worth note the differences in estimates for adjustment costs and the inverse of Frisch elasticity. The dataset with expectations increases the estimate of adjustment costs but substantially decreases persistence (and increases the standard deviation) of the investment-specific technological shocks. These conclusions are also valid are not exclusive of the full information model. The same results apply when comparing the ICK models for both datasets (columns 2 and 4 from Table 3.2).

Comparing the parameter estimates for full and dispersed information model highlights the role of the informational frictions. Most of the parameters related to nominal and real frictions are relatively stable after the inclusion of informational frictions. This implies that such frictions are complementary to the standard frictions in DSGE models.

The dataset including expectations is very informative for the noise-to-signal ratios. The change in parameter estimates for both datasets reflects such information. Informational frictions are particularly strong for the productivity and moderate for monetary and expenditure shocks. The other shocks have rather small noise-to-signal ratios, which imply that for such shocks the responses such be similar to the full informational counterpart (given that other parameters are the same, which is not the case).

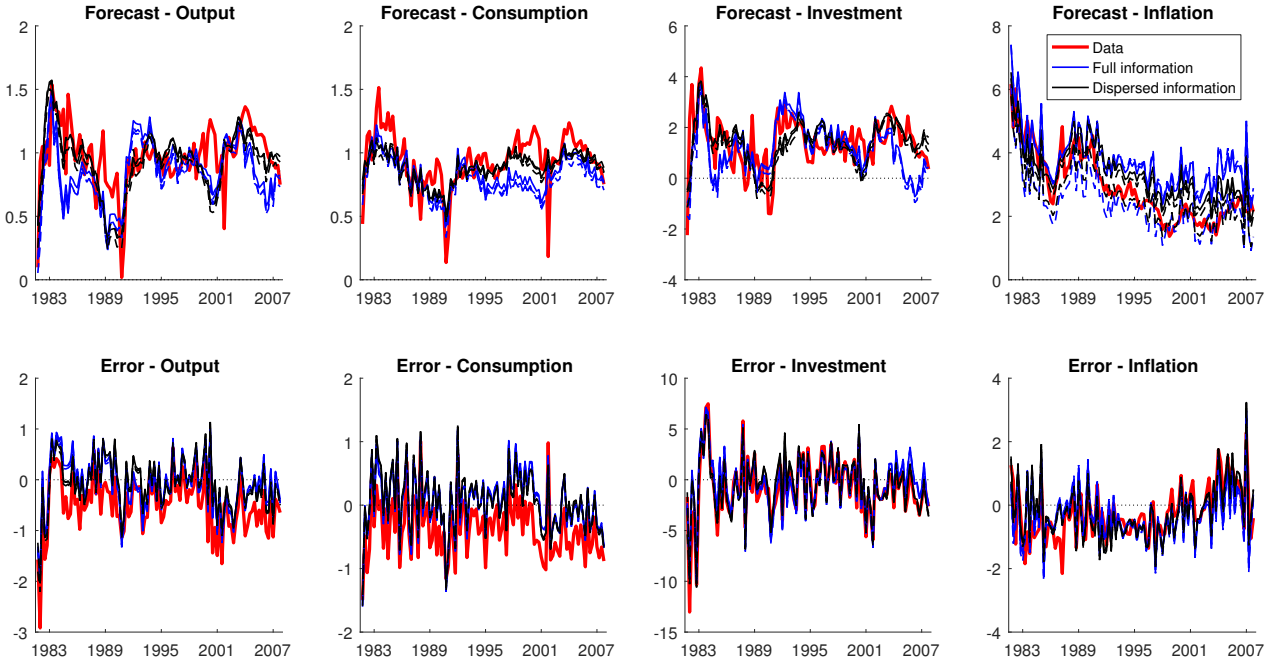
Model Comparison

This section compares the model fit regarding the expectations data. A straightforward way to compare is to plot the model implied expectations for both models and compare with the data. The model implied expectations is computed using the Kalman smoother. If there were no measurement errors, the models would perfectly fit the forecast error data. Figure 3.1 does this comparison for both the forecast and the forecast error for output, investment consumption and inflation. For the model implied variables, the solid line and dotted lines represent the mean and the 90% bands of the predictive distribution of such variables.

By inspection, one can see that the expectations of the Dispersed information model track better the forecast data (the forecast error data is more erratic and harder to see). This is true in particular for the consumption forecast during the 90's and the investment forecast during the 2000's. Both models have some difficult to track inflation expectations, but the ICK model is relatively closer.

For a more straightforward comparison, Table 3.3 computes the correlation between model implied and SPF data for the forecast and forecast error in level and in first-difference.

Figure 3.1: Model implied expectations and SPF data



Regarding the forecast error, the correlation is very similar for both models. When it comes to the forecast data, the Dispersed information model has a better fit under this measure for the level. Moreover, both models do not track well the first-difference of the forecast data.

Table 3.3: Correlation between model implied expectations and SPF data

	Forecast				Forecast error			
	Output	Consumption	Investment	Inflation	Output	Consumption	Investment	Inflation
Full information								
Level	0.39	0.56	0.45	0.80	0.90	0.94	0.92	0.67
Difference	0.11	0.01	0.25	0.19	0.94	0.96	0.96	0.77
Dispersed information								
Level	0.56	0.64	0.57	0.85	0.92	0.94	0.94	0.76
Difference	0.11	0.00	0.25	0.21	0.95	0.97	0.97	0.81

A formal manner to compare the fit of the each model m, $\mathcal{M}_m$. is to compute the marginal likelihood

$$p(\mathcal{Y}|\mathcal{M}_m) = \int p(\mathcal{Y}|\mathcal{M}_m, \Theta_m) p(\Theta_m|\mathcal{M}_m) d\Theta_m \tag{3.95}$$

where $\mathcal{Y}$ denotes the dataset including the expectation data and Θ_m, $p(\mathcal{Y}|\mathcal{M}_m, \Theta_m)$ is the

of model m and $p(\Theta_i|\mathcal{M}_m)$ denote the estimated parameters, likelihood function and prior distribution of model m, respectively. Intuitively, the marginal likelihood integrates out the influence of the parameter estimates on the posterior distribution. It is a measure of how well the model fits the data.

Following Del Negro and Eusepi (2011), consider that $\mathcal{Y} = \{\mathcal{Y}_0, \mathcal{Y}_1\}$, where $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote the macroeconomic and expectation data, respectively. One of the goals is to evaluate how well each model fits the expectation data. Such comparison can be done by computing the conditional marginal likelihood given by

$$p(\mathcal{Y}|\mathcal{Y}_0, \mathcal{M}_m) = \int p(\mathcal{Y}|\mathcal{Y}_0, \mathcal{M}_m, \Theta_m)p(\Theta_m|\mathcal{Y}_0, \mathcal{M}_m)d\Theta_m. \tag{3.96}$$

$p(\mathcal{Y}|\mathcal{Y}_0, \mathcal{M}_m)$ measures how well the model m explain the dataset $\mathcal{Y}$, conditional on the data $\mathcal{Y}_0$. In other words, it measures how well the model fits the remaining data $\mathcal{Y}_1$, provided that the model is estimated using information from both datasets $\mathcal{Y}_0$ and $\mathcal{Y}_1$.

Using the Bayes rule, Del Negro and Eusepi (2011) shows that such object can be computed by simply taking the difference of the marginal likelihood for the dataset $\mathcal{Y}_0$ from the marginal likelihood for the dataset, $\mathcal{Y}$, i.e, $p(\mathcal{Y}|\mathcal{Y}_0, \mathcal{M}_m) = p(\mathcal{Y}|\mathcal{M}_m) - p(\mathcal{Y}_0|, \mathcal{M}_m)$. One alternative would be to compute the marginal likelihood only in the expectation data, $\mathcal{Y}_1$. However, it is unlikely the only the expectations provide sufficient information for estimating all parameter accurately.

Table 3.4 shows the log of the marginal likelihood for both models and both datasets. Following Geweke (1999), the marginal likelihood is computed as the modified harmonic mean over the posterior draws. By comparing the marginal likelihood of the two competing models for the dataset with macroeconomic data only, there is a difference of roughly 85 log points for the full information model (log marginal likelihood closer to zero means better fit). This implies that the latter fits substantially better than the dispersed information model.

Table 3.4: Model comparison

	Macroeconomic data only (1)	Macroeconomic and Expectations data (2)	Difference (2) - (1)
Full information	-904.95	-1309.50	-404.55
Dispersed information	-989.72	-1333.30	-343.58
Difference (Dispersed - Full)	-84.77	-23.80	60.97

Note: Following Geweke (1999), the marginal likelihood is computed as the modified harmonic mean over the posterior remainning 200.000 draws after the burn-in.

A similar pattern applies for the complete dataset. The full information model has an

advantage of roughly 24 log points. This implies in very strong odds in comparison with the dispersed information model. However, the third column of Table 3.4 shows the difference in the marginal likelihood for both models. This computes the conditional marginal likelihood from equation (3.96). In this case, there is a roughly 61 log points margin in favor of the dispersed information model.

In other words, the full information model is substantially better than the ICK model for explaining the macroeconomic data. When it comes to the dataset complete dataset, it also does a better job than the ICK model, but by a much weaker margin. By taking the difference of the two marginal likelihoods, it makes clear that the ICK model explain much better the expectation data alone despite the (relative) poor job regarding fitting the macroeconomic data.

In a nutshell, what do those comparisons imply? They reflect that the full information model indeed fits better both datasets than the dispersed information model. However, when it comes to *only* the expectation data, the dispersed information model does a better job. It comes at a cost that it has a worse fit in the macroeconomic data.

The comparison with Del Negro and Eusepi (2011) is insightful. They do a similar comparison that is different in two aspects. First, their dataset includes only inflation expectation data (4-quarter ahead instead of 1-quarter). Second, the comparison is between the full information and an imperfect information model, which firms cannot disentangle monetary from target shocks. The uncertainty in that paper is simpler because it applies only to two shocks and faced by only firms under imperfect but common information. Surprisingly, they find that the perfect information model fits better even expectation data alone. Therefore, our results imply that introducing richer informational frictions is a valuable venue for explaining expectation data.

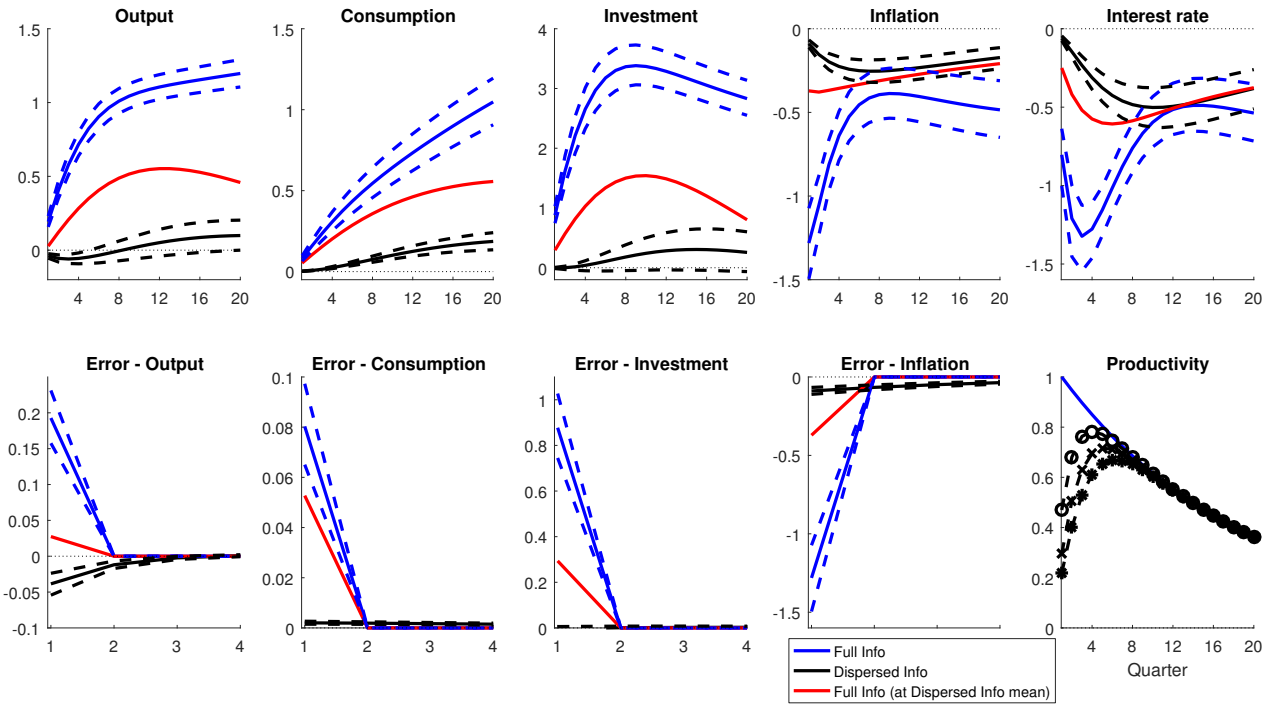
Impulse response functions

This subsection compares the dynamic properties of both models after the realization of shocks. I will focus on the productivity and monetary shock for two reasons. First, they are among the shocks that receive more emphasis in the literature. Second, it turns out that they are the ones that have larger informational frictions by the estimates of Table 3.2.

Figure 3.2 shows the impulse response functions for the productivity shock. The blue and black, solid and dotted lines, refer to the mean and 90% HPD interval for the full and dispersed information models, respectively. The solid red line is included for comparison. It shows the IRF that would occur in the full information model for the parameters estimated at the dispersed information model. This comparison helps to understand the whether the

differences in dynamics result from different parameter estimates of the common parameters or because of the informational frictions. Finally, the last panel shows the realization of the shock in solid blue and the higher-order expectations up to the third order in marked dashed black lines ("$\circ$", "$\times$" and "$*$" markers denote the first, second and third orders, respectively.)

Figure 3.2: Impulse response function: Productivity shock



Note: The periods are in quarters. For forecast errors only the first 4 quarters are shown for convenience.

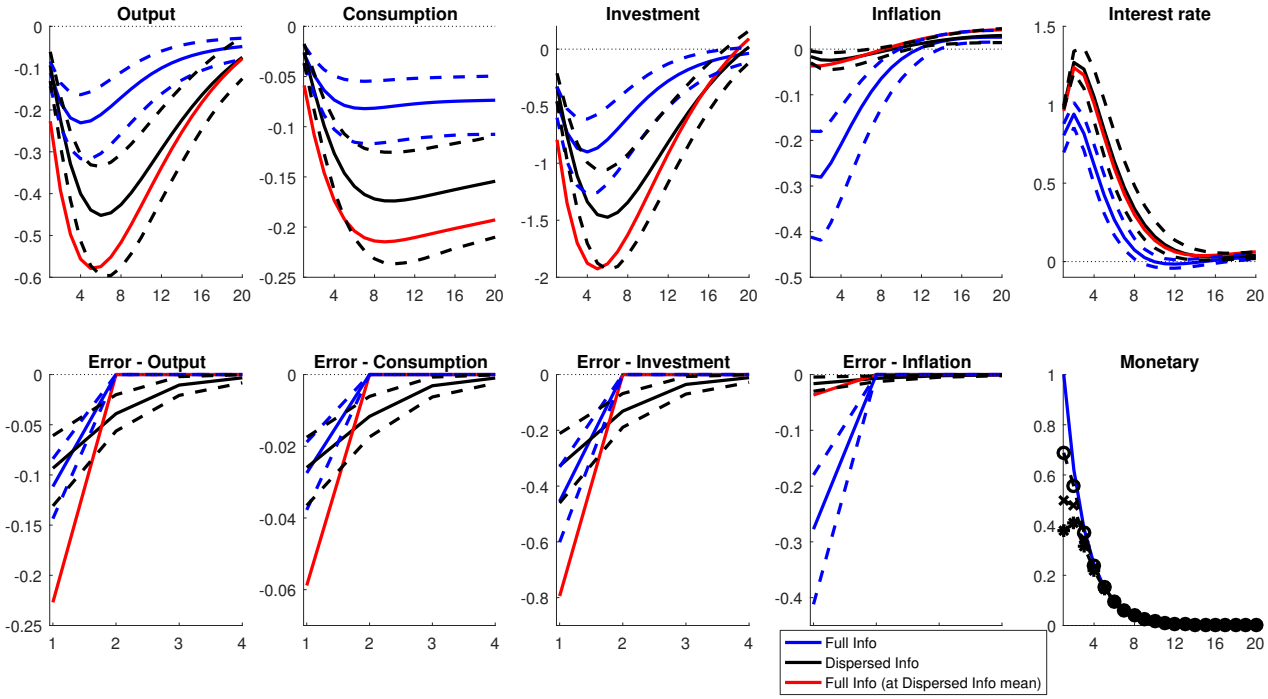
The difference in the dynamics of the full information and dispersed information models is remarkable. The dispersed information implies more subtle and sluggish response to the productivity shock. The solid red lines make it clear that this is partly owing to stronger nominal and real rigidity in the estimation of the dispersed information model. Informational frictions also play an essential role: it takes roughly two years to agents, on average, to learn about the full extent of the shock. This implies that agents decisions are made based on expectations about the productivity shock that are substantially lower than the actual shock. This gradual learning explains the somewhat muted response to the productivity shock.

It is also important to note the difference of the dynamics of the forecast errors. Under full information, the forecast errors respond only the period of the shock and instantly goes back to steady-state afterwards. Under dispersed information, the learning process induces autocorrelated forecast errors. For the productivity shock, they are persistent but rather

small.

Figure 3.3 shows the same exercise for the monetary shock. By comparing the blue with the solid red lines, it is clear that most of the differences of the full and dispersed information models are due to the different parameter estimates. High price rigidity for the ICK model implies a more muted response of inflation and stronger real effects. Since the monetary shock is relatively moderate informational frictions are moderate for this shock (point estimates of r_r are roughly 0.6), they play a relatively minor role. Agents' expectations react more on impact and learn about the shock entirely within a year. This diminishes the real effects as agents based on expectations that are below the actual shock.

Figure 3.3: Impulse response function: Monetary shock



Note: The periods are in quarters. For forecast errors only the first 4 quarters are shown for convenience.

For the monetary shock, the property of autocorrelated forecast errors is more pronounced. Therefore, stronger informational frictions about one particular shock do not imply that forecast errors after such shock are more persistent. This is true because, in the ICK model, nominal and real frictions also matter. Intuitively, since agents do not know current endogenous variables, their current forecast error translates to forecast errors in the one-quarter-ahead expectations due to the endogenous persistence. In the case of the full information model, the forecast error is uncorrelated because firms observe the current

endogenous variables.

3.4 Concluding remarks

This paper proposes a tractable solution method for a general class of DSGE modes with imperfect common knowledge. The solution extends previous methods by including endogenous state variables into the system of linear rational expectation conditions.

The algorithm to solve such models pushes the quantitative frontier for imperfect and dispersed information models by allowing to introduce such informational structure to medium-scale DSGE models. As a primer exercise, I explore the ability of informational frictions to explain the macroeconomic and expectations data. The model under imperfect and dispersed information explain better the expectation data whereas the full information model explains better the macroeconomic aggregates. Moreover, the estimates suggest that informational frictions are not substitutes of the standard real and nominal frictions.

The solution method opens the opportunity for future research to evaluate a wide range of informational frictions as well other types of shocks suggested in the literature such as news and noise shocks, "confidence" shocks and many others, for business cycle fluctuations and economic policy. As suggested by Angeletos et al. (forthcoming), this is a possible alternative to improve the fit of macroeconomic aggregates for the imperfect information model.

3.A Proofs

3.A.1 Proof of Lemma 3.2

The Kalman filter delivers the individual conditional expectation, $E_{it}[\cdot] \equiv E[\cdot|\mathcal{I}_t^i]$, where $\mathcal{I}_t^i = \{S_{i\tau}, \tau \leq t\}$ is the information set of individual i in period t.

The state equation is the hierarchy of expectations that is given by the state equation (3.8), restated for convenience:

$$x_t^{(0:\bar{k})} = Ax_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t, \tag{3.A.1}$$

and the agent i with the observational equation (3.20), also restated:

$$S_{it} = Cx_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t, \tag{3.A.2}$$

Each agent i uses the Kalman filter and find the update equation given by

$$E_{i,t}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Z_{it} - E_{i,t-1}\left[Z_{it}\right]\right], \tag{3.A.3}$$

where K_t is the Kalman gain given by

$$K_t = \left[P_{t/t-1}C' + \Sigma_{cov}\right]\left[CP_{t/t-1}C' + \Sigma_{obs}\right]^{-1}, \tag{3.A.4}$$

where $\Sigma_{obs} = D_v\Sigma_v D_v' + D_u\Sigma_u D_u'$ and and $\Sigma_{cov} = B_u\Sigma_u D_u'$. As usual, the mean squared error (MSE) of the one-period ahead prediction error is given by

$$P_{t+1/t} = A\left[P_{t/t-1} - K_t\left[CP_{t/t-1} + \Sigma_{cov}\right]\right]A' + \Sigma_{state}, \tag{3.A.5}$$

where $\Sigma_{state} = B_\varepsilon\Sigma_\varepsilon B_\varepsilon' + B_u\Sigma_u B_u'$. For details of this deviation, see for instance Hamilton (1995, chap. 13).

Using the observational equation, (3.A.2), taking expectations and inserting in (3.A.3) one can find:

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Cx_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t - CE_{i,t-1}\left[x_t^{(0:\bar{k})}\right]\right] \tag{3.A.6}$$

Therefore, one can rewrite the equation above as

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - K_tC\right)E_{it-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Cx_t^{(0:\bar{k})} + D_v v_{it} + D_u u_t\right] \tag{3.A.7}$$

where $k = n(\bar{k}+1)$. Using the fact that $E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] = AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right]$ and substituting equation (3.A.1), one can find:

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - K_tC\right)AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + K_tCAx_{t-1}^{(0:\bar{k})} + K_tC(B_\varepsilon\varepsilon_t + B_u u_t) + K_t[D_v v_{it} + D_u u_t] \tag{3.A.8}$$

I follow the literature by focusing in the stationary equilibrium. Therefore, the expectation of each individual i in the stationary equilibrium is the one which the MSE is in steady-state, i.e., firms update their forecast based on the steady-state Kalman gain. In other words, the dynamics of expectations depends only in the properties of the process they are forecasting and do not depend in the period t. Using equations for $P_{t+1/t}$, $P_{t/t}$ and K_t one can find the Riccatti equation

$$P_{t+1/t} = A\left[P_{t/t-1} - P_{t/t-1}C'\left[CP_{t/t-1}C' + \Sigma_{obs}\right]^{-1}\left[CP_{t/t-1} + \Sigma_{cov}\right]\right]A' + \Sigma_{state} \tag{3.A.9}$$

Therefore, one need to iterate this equation to find the steady-state MSE, $\bar{P}$, and compute its counterpart Kalman gain, $\bar{K}$. Nimark (2017) shows that if is x_t stationary process, then the expectations hierarchy about this process, $x_t^{(0:\bar{k})}$, is also stationary. This and the fact that Σ_ε is positive definite, then there exists a steady-state solution such that $\bar{P} = P_{t+1|t} = P_{t|t-1}$ which implies the steady-state Kalman gain $\bar{K} = K_t = K_{t-1}$ (see Hamilton; 1995, chap. 13).

The individual expectation in equation of Proposition 3.2 is one in equation (3.A.8) above using the steady-state Kalman gain, $\bar{K}$. Moreover, the average expectation is easily computed by

$$\begin{aligned} E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] &\equiv \int_0^1 E_{it}\left[x_t^{(0:\bar{k})}\right]di \\ &= \left(I_k - \bar{K}C\right)AE_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{K}CAx_{t-1}^{(0:\bar{k})} + \bar{K}CB_\varepsilon\varepsilon_t + \bar{K}(CB_u + D_u)u_t \end{aligned} \tag{3.A.10}$$

□

3.A.2 Proof of Lemma 3.4

In this subsection, following Nimark (2015), I derive the Kalman filter recursions when the observational equations depends on the lag of the state-variable. Then, I show the dynamics of individual expectations and first-order expectations of $x_t^{(0:\bar{k})}$.

For convenience, I rewrite the state equation and the observational equations:

$$x_t^{(0:\bar{k})} = Ax_{t-1}^{(0:\bar{k})} + B_\varepsilon\varepsilon_t + B_u u_t, \tag{3.A.11}$$

$$Z_{it} = C_1 x_t^{(0:\bar{k})} + C_2 x_{t-1}^{(0:\bar{k})} + Nw_{t-1} + \hat{D}_v v_{it} + \hat{D}_u u_t, \tag{3.A.12}$$

where w_{t-1} is a vector of predetermined variables. By the Projection Theorem, we can write:

$$E_{i,t}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[Z_{it} - E_{i,t-1}\left[Z_{it}\right]\right] \tag{3.A.13}$$

where $K_t = E\left[\tilde{x}_t^{(0:\bar{k})}\tilde{Z}_{it}{}'\right] E\left[\tilde{Z}_{it}\tilde{Z}_{it}{}'\right]^{-1}$ and $\tilde{\zeta}_t \equiv \zeta_t - E_{i,t-1}[\zeta_t]$ for any variable ζ_t.

Using equation (3.A.11) and taking its expectation, one can compute $\tilde{x}_t^{(0:\bar{k})}$ as

$$\tilde{x}_t^{(0:\bar{k})} = A\tilde{x}_{t-1}^{(0:\bar{k})} + B_\varepsilon \varepsilon_t + B_u u_t. \tag{3.A.14}$$

The same can be applied to equation (3.A.12) to find $\tilde{Z}_{it}$ such that

$$\tilde{Z}_{it} = (C_1 A + C_2)\tilde{x}_{t-1}^{(0:\bar{k})} + C_1 B_\varepsilon \varepsilon_t + (C_1 B_u + \hat{D}_u)u_t + \hat{D}_v v_{it}. \tag{3.A.15}$$

Therefore, the first term of K_t can be computed by

$$\begin{aligned} E\left[\tilde{x}_t^{(0:\bar{k})}\tilde{Z}_{it}{}'\right] &= AE\left[\left(x_{t-1}^{(0:\bar{k})} - E_{i,t-1}(x_{t-1}^{(0:\bar{k})})\right)\left(x_{t-1}^{(0:\bar{k})} - E_{i,t-1}(x_{t-1}^{(0:\bar{k})})\right)'\right](C_1 A + C_2)' \\ &\quad + B_\varepsilon E\left[\varepsilon_t \varepsilon_t'\right](C_1 B_\varepsilon)' + B_u E\left[u_t u_t'\right](C_1 B_u + \hat{D}_u)' \\ &= AP_{t-1|t-1}(C_1 A + C_2)' + \Sigma_{state} C_1' + \Sigma_{cov}, \end{aligned} \tag{3.A.16}$$

where in the first equality, I used that $\tilde{x}_{t-1}^{(0:\bar{k})}$, ε_t, v_{it} and u_t are uncorrelated and the second equality used the definitions, $\Sigma_{state} = B_\varepsilon \Sigma_\varepsilon B_\varepsilon' + B_u \Sigma_u B_u'$ and $\Sigma_{cov} = B_u \Sigma_u \hat{D}_u'$

Moreover, the second term of K_t can be computed by

$$\begin{aligned} E\left[\tilde{Z}_{it}\tilde{Z}_{it}'\right] &= (C_1 A + C_2)E\left[\left(x_{t-1}^{(0:\bar{k})} - E_{i,t-1}(x_{t-1}^{(0:\bar{k})})\right)\left(x_{t-1}^{(0:\bar{k})} - E_{i,t-1}(x_{t-1}^{(0:\bar{k})})\right)'\right](C_1 A + C_2)' \\ &\quad + C_1 B_\varepsilon E\left[\varepsilon_t \varepsilon_t'\right](C_1 B_\varepsilon)' + (C_1 B_u + \hat{D}_u)E\left[u_t u_t'\right](C_1 B_u + \hat{D}_u)' + \hat{D}_v E\left[v_{it} v_{it}'\right]\hat{D}_v' \\ &= (C_1 A + C_2)P_{t-1|t-1}(C_1 A + C_2)' + C_1 \Sigma_{state} C_1' + C_1 \Sigma_{cov} + \Sigma_{cov}' C_1' + \Sigma_{obs} \end{aligned} \tag{3.A.17}$$

where in the first equality, I used again that $\tilde{x}_{t-1}^{(0:\bar{k})}$, ε_t, v_{it} and u_t are uncorrelated and the last equality used the definition $\Sigma_{obs} = \hat{D}_u \Sigma_u \hat{D}_u' + \hat{D}_v \Sigma_v \hat{D}_v'$.

Therefore, the Kalman gain is computed can be computed by

$$\begin{aligned} K_t = &\left[AP_{t-1|t-1}(C_1 A + C_2)' + \Sigma_{state} C_1' + \Sigma_{cov}\right] \\ &\left[(C_1 A + C_2)P_{t-1|t-1}(C_1 A + C_2)' + C_1 \Sigma_{state} C_1' + C_1 \Sigma_{cov} + \Sigma_{cov}' C_1' + \Sigma_{obs}\right]^{-1}. \end{aligned} \tag{3.A.18}$$

Now, I need to find the expression for the mean square error of one-period ahead forecast of the state variable, $P_{t+1/t}$, such that

$$
\begin{aligned}
P_{t+1/t} &= E\left[\left(x_{t+1}^{(0:\bar{k})} - E_{i,t}(x_{t+1}^{(0:\bar{k})})\right)\left(x_{t+1}^{(0:\bar{k})} - E_{i,t}(x_{t+1}^{(0:\bar{k})})\right)'\right] \\
&= E\left[\left(A(x_t^{(0:\bar{k})} - E_{i,t}(x_t^{(0:\bar{k})})) + B_\varepsilon \varepsilon_t + B_u u_t\right)\left(A(x_t^{(0:\bar{k})} - E_{i,t}(x_t^{(0:\bar{k})})) + B_\varepsilon \varepsilon_t + B_u u_t\right)'\right] \\
&= AP_{t/t}A' + \Sigma_{state}
\end{aligned}
\tag{3.A.19}
$$

where the second inequality uses equation (3.A.11) and the fact that $E_{i,t}(x_{t+1}^{(0:\bar{k})}) = AE_{i,t}(x_t^{(0:\bar{k})})$. Analogously,

$$
\begin{aligned}
P_{t/t} &= E\left[\left(x_t^{(0:\bar{k})} - E_{i,t}(x_t^{(0:\bar{k})})\right)\left(x_t^{(0:\bar{k})} - E_{i,t}(x_t^{(0:\bar{k})})\right)'\right] \\
&= E\left[\left(x_t^{(0:\bar{k})} - E_{i,t-1}(x_t^{(0:\bar{k})}) - K_t\tilde{Z}_{it}\right)\left(x_t^{(0:\bar{k})} - E_{i,t-1}(x_t^{(0:\bar{k})}) - K_t\tilde{Z}_{it}\right)'\right] \\
&= E\left[\tilde{x}_t^{(0:\bar{k})}\left(\tilde{x}_t^{(0:\bar{k})}\right)'\right] - E\left[\tilde{x}_t^{(0:\bar{k})}\tilde{Z}_{it}'\right]K_t' - K_tE\left[\tilde{Z}_{it}(\tilde{x}_t^{(0:\bar{k})})'\right] + K_tE\left[\tilde{Z}_{it}\tilde{Z}_{it}'\right]K_t' \\
&= P_{t/t-1} - K_t\left[AP_{t|t-1}(C_1A + C_2)' + \Sigma_{state}C_1' + \Sigma_{cov}\right]'
\end{aligned}
\tag{3.A.20}
$$

where the second equality uses the Projection Theorem and the fourth equality uses the fact that $K_tE\left[\tilde{Z}_{it}\tilde{Z}_{it}'\right]K_t' = E\left[\tilde{x}_t^{(0:\bar{k})}\tilde{Z}_{it}'\right]K_t'$.

Given the derivation of K_t and $P_{t+1|t}$, I can turn to the expressions for expectations. Using the observational equation, (3.A.2), taking expectations and inserting in (3.A.13) one can find:

$$
E_{it}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[C_1x_t^{(0:\bar{k})} + C_2x_{t-1}^{(0:\bar{k})} + \hat{D}_v v_{it} + \hat{D}_u u_t - C_1E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] - C_2E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right]\right.
\tag{3.A.21}
$$

Rearranging and using the fact that $E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] = AE_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right]$, one can rewrite the equation above as

$$
E_{it}\left[x_t^{(0:\bar{k})}\right] = \left[(I_k - K_tC_1)A - K_tC_2\right]E_{it-1}\left[x_t^{(0:\bar{k})}\right] + K_t\left[C_1x_t^{(0:\bar{k})} + C_2x_{t-1}^{(0:\bar{k})} + \hat{D}_v v_{it} + \hat{D}_u u_t\right]
\tag{3.A.22}
$$

where $k = n(\bar{k}+1)$. Substituting equation (3.A.1), one can find:

$$
\begin{aligned}
E_{it}\left[x_t^{(0:\bar{k})}\right] = &\left[\left(I_k - K_t C_1\right) A - K_t C_2\right] E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + K_t\left[C_1 A + C_2\right] x_{t-1}^{(0:\bar{k})} \\
&+ K_t C_1 (B_\varepsilon \varepsilon_t + B_u u_t) + K_t[\hat{D}_v v_{it} + \hat{D}_u u_t]
\end{aligned} \tag{3.A.23}
$$

again, I follow the literature by focusing in the stationary equilibrium. Therefore, the expectation of each individual i in the stationary equilibrium is the one which the MSE is in steady-state, i.e., firms update their forecast based on the steady-state Kalman gain. In other words, the dynamics of expectations depends only in the properties of the process they are forecasting and do not depend in the period t. Using equations for $P_{t+1/t}$, $P_{t/t}$ and K_t one can find the Riccatti equation

$$
\begin{aligned}
P_{t/t} = &A P_{t-1/t-1} A' + \Sigma_{state} - \left[A P_{t-1|t-1}(C_1 A + C_2)' + \Sigma_{state} C_1' + \Sigma_{cov}\right] \\
&\left[(C_1 A + C_2) P_{t-1|t-1}(C_1 A + C_2)' + C_1 \Sigma_{state} C_1' + C_1 \Sigma_{cov} + \Sigma_{cov}' C_1' + \Sigma_{obs}\right]^{-1} \\
&\left[A\left(A P_{t-1/t-1} A' + \Sigma_{state}\right)(C_1 A + C_2)' + \Sigma_{state} C_1' + \Sigma_{cov}\right]'
\end{aligned} \tag{3.A.24}
$$

Note that in this version of the Kalman filter, it is easier to find the recursions in terms of $P_{t|t}$ and $P_{t-1|t-1}$. The standard procedure for the standard Kalman filter is to find the recursions in terms of $P_{t+1|t}$ and $P_{t|t-1}$ as in Appendix 3.A.1. This difference is immaterial since it is the Kalman gain that is needed to compute expectations.

Therefore, one need to iterate this equation to find the steady-state MSE, $\hat{P}$, and compute its counterpart Kalman gain, $\hat{K}$. By the same reasoning from Appendix 3.A.1, there exists a steady-state $\hat{P}$.

The individual expectation in equation of Lemma 3.4 is one in equation (3.A.8) above using the steady-state Kalman gain, $\hat{K}$.

Moreover, the first order expectation is easily computed by

$$
\begin{aligned}
E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] &\equiv \int_0^1 E_{it}\left[x_t^{(0:\bar{k})}\right] di \\
&= \left[\left(I_k - \hat{K} C_1\right) A - \hat{K} C_2\right] E_{t-1}^{(1)}\left[x_{t-1}^{(0:\bar{k})}\right] + \hat{K}\left[C_1 A + C_2\right] x_{t-1}^{(0:\bar{k})} \\
&+ \hat{K} C_1 B_\varepsilon \varepsilon_t + \hat{K}(C_1 B_u + \hat{D}_u) u_t
\end{aligned} \tag{3.A.25}
$$

□

3.B Dataset description

Data on macroeconomic variables are from FRED and expectation data are from Survey of Professional Forecasters (SPF). (FRED and SPF mnemonics are in parenthesis). Real GDP (GDPC1), real consumption (PCECC96), real investment (GPDIC1) are divided by the smoothed civilian noninstitutional population index (CNP16OV) to get per capita variables consistent with the model. Following Edge and Gürkaynak (2010), I use HP filtered index of population to correct for spikes in growth rate from adjustments of measurement from the Census. Real wage is computed by dividing the Compensation in the nonfarm business sector (PRS85006063) by the GDP deflator (GDPDEF), which is also used as observable. Per capita total hours is computed by dividing the hours of all persons in nonfarm business sector (HOANBS) from the smoothed population index. The nominal interest rate is computed by the log of the gross federal funds rate (FF) to be consistent with the log-linearization procedure.

$$
\begin{aligned}
Y_t^{data} &= GDPC1_t/HP(CNP16OV_t) \\
C_t^{data} &= PCECC96_t/HP(CNP16OV_t) \\
I_t^{data} &= GPDIC1_t/HP(CNP16OV_t) \\
W_t^{data} &= PRS85006063_t/GDPDEF_t \\
d_{y,t}^{data} &= 100log(Y_t^{data}/Y_{t-1}^{data}) \\
d_{c,t}^{data} &= 100log(C_t^{data}/C_{t-1}^{data}) \\
d_{i,t}^{data} &= 100log(I_t^{data}/I_{t-1}^{data}) \\
d_{w,t}^{data} &= 100log(W_t^{data}/W_{t-1}^{data}) \\
L_t^{data} &= HOANBS_t/HP(CNP16OV_t) \\
\Pi_t^{data} &= 400log(GDPDEF_t/GDPDEF_{t-1}) \\
R_t^{data} &= 100log(1+FF_t/100)
\end{aligned}
$$

where $RINVEST3_t = RNRESIN3_t + RRESINV3_t + RCBI3_t$ and $RINVEST2_t = RNRESIN2_t + RRESINV2_t + RCBI2_t$

The expectation data consists in the current and 1-quarter ahead expectation from real GDP (RGDP2 and RGDP3) and GDP deflator (PGDP2 and PGDP3), real consumption (RCONSUM3 and RCONSUM2) and real investment. In the SPF, expectations on investment are done by its three components consistently with definitions in national accounts. Therefore, real investment expectation for the current and 1-quarter ahead (which I define RINVEST3 and RINVEST2) is obtained by summing the expectations about real nonresidential fixed

investment (RNRESIN3 and RNRESIN2), real residential fixed investment (RRESINV3 and RRESINV2) and real change in private inventories (RCBI3 and RCBI2).

The forecast errors are constructed by taking the difference of the observed data on period t with the expectation of the same data in period $t-1$.

$$
\begin{aligned}
Exp^{data}_{y,q=1,t} &= RGDP3_t/HP(CNP16OV_{t+1}) \\
Exp^{data}_{y,q=0,t} &= RGDP2_t/HP(CNP16OV_t) \\
Exp^{data}_{c,q=1,t} &= RCONSUM3_t/HP(CNP16OV_{t+1}) \\
Exp^{data}_{c,q=0,t} &= RCONSUM2_t/HP(CNP16OV_t) \\
Exp^{data}_{i,q=1,t} &= RINVEST3_t/HP(CNP16OV_{t+1}) \\
Exp^{data}_{i,q=0,t} &= RINVEST2_t/HP(CNP16OV_t)
\end{aligned}
$$

$$
\begin{aligned}
Exp^{data}_{dy,q=1,t} &= 100log(Exp^{data}_{y,q=1,t}/Exp^{data}_{y,q=0,t}) \\
Exp^{data}_{dc,q=1,t} &= 100log(Exp^{data}_{c,q=1,t}/Exp^{data}_{c,q=0,t}) \\
Exp^{data}_{di,q=1,t} &= 100log(Exp^{data}_{i,q=1,t}/Exp^{data}_{i,q=0,t}) \\
Exp^{data}_{\pi,q=1,t} &= 100log(PGDP3_t/PGDP2_t) \\
Error^{data}_{dy,q=1,t} &= d^{data}_{y,t} - Exp^{data}_{dy,q=1,t-1} \\
Error^{data}_{dc,q=1,t} &= d^{data}_{c,t} - Exp^{data}_{dc,q=1,t-1} \\
Error^{data}_{di,q=1,t} &= d^{data}_{i,t} - Exp^{data}_{di,q=1,t-1} \\
Error^{data}_{\pi,t} &= \Pi^{data}_{t} - Exp^{data}_{\pi,q=1,t-1}
\end{aligned}
$$

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