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ESCOLA BRASILEIRA DE ECONOMIA E FINANÇAS

LUCIANO FABIO BUSATTO VENTURIM

Labor Demand in Relational Contracts with Private Information

RIO DE JANEIRO
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Labor Demand in Relational Contracts with Private Information

Dissertação para obtenção do grau de mestre apresentada à Escola Brasileira de Economia e Finanças

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LUCIANO FABIO BUSATTO VENTURIM

"LABOR DEMAND IN RELATIONAL CONTRACTS WITH PRIVATE INFORMATI''.

DISSERTAÇÃO APRESENTADO(A) AO CURSO DE MESTRADO EM ECONOMIA PARA OBTENÇÃO DO GRAU DE MESTRE(A) EM ECONOMIA.

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ASSINATURA DOS MEMBROS DA BANCA EXAMINADORA

PRESIDENTE DA COMISSÃO EXAMINADORA: PROFº/ª LUCAS JÓVER MAESTRI

PROFº/ª LUCAS JÓVER MAESTRI
ORIENTADOR(A)

PROFº/ª CARLOS EUGENIO ELLERY DA COSTA
MEMBRO

PROFº/ª DONG WEI
MEMBRO

RIO DE JANEIRO, 28 DE MARÇO DE 2023.

PROFº/ª LUCAS JÓVER MAESTRI
COORDENADOR(A)

PROFº ANTONIO DE ARAUJO FREITAS JUNIOR
PRÓ-REITOR DE ENSINO, PESQUISA E PÓS-GRADUAÇÃO

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Resumo

Consideramos um ambiente de contratação relacional com informação privada em que uma firma também escolhe o número de funcionários. O modelo nos permite analisar o *trade-off* entre as margens extensiva e intensiva. Mostramos que sempre que o esforço individual é distorcido para baixo, o número de funcionários pode até ser distorcido para cima, dependendo das funções de produção e custo. Quando a produtividade cai, a empresa demite alguns funcionários enquanto aumenta o esforço exercido pelos restantes. Com informação privada, a demissão inicial é maior, mas a empresa começa a contratar novamente em períodos subsequentes, com o número de funcionários se aproximando do ótimo no longo prazo. Os resultados sugerem que, nesse cenário, o esforço é distorcido apenas devido a problemas de comprometimento, enquanto o número de funcionários também é distorcido devido a assimetrias informacionais.
Palavras-chave: Contratos Relacionais, Informação Privada, Demanda de Trabalho

Abstract

We consider a relational contracting setting with private information in which a firm also chooses the number of employees. The model allows us to analyze the trade-off between the extensive and intensive margins. We show that whenever individual effort is distorted downward, the number of employees may even be distorted upward, depending on the production and cost functions. When productivity falls, the firm lays off some employees while increasing the effort exerted by the remaining ones. With private information, the initial layoff is bigger, but the firm starts hiring again in subsequent periods, with the number of employees approaching first-best in the long run. The results suggest that, in this setting, effort is distorted only due to commitment issues, while the number of employees is also distorted due to informational asymmetries.

Keywords: Relational Contracts, Private Information, Labor Demand

Contents

1 Introduction

In many economic settings, legislative restrictions or unverifiable outcomes may hinder one's ability to commit to formal, long-term contracts. This lack of commitment constrains efficiency by creating opportunities for ex-post deviations that cannot be punished beyond the deviating party's outside option. To align incentives, a firm may link payments to the effort each worker exerts. However, if the firm cannot formally commit to bonus payments for the aforementioned reasons, no worker would exert effort anticipating breaches in the agreement, and, consequently, the firm would (optimally) not employ anyone. Yet, in a dynamic setting, present actions can be punished or rewarded with future payoffs generated by the ongoing relationship: the more valuable the relationship, the less incentive the firm has to renege on the conditional payments, so production is less inefficient. Changes in the environment might affect how much the firm values the relationship with its current employees and, consequently, how committed the firm is. Economic recessions, technological changes implemented by competitors, and new legislation are some examples of external disruptions that might influence, temporarily or not, the manager's decisions (assumed to be completely aligned with the shareholders) about how many workers to employ and how to incentivize them given a new set of market conditions. Generally, recessions may create the need, unique to each firm, for layoffs and reductions in the variable pay of employees. This means that if the manager (she) has private information on the value of a worker to the firm, she might be incited to announce, ex-post, a bleak future and pay the employees less than she would otherwise.

A large body of literature on relational contracts has sought to understand how this lack of commitment affects contracts and decisions and how efficient they can be in different settings and when paired with other externalities such as asymmetric information. Most of them focused on the relationship between a principal and a unique agent, or a fixed number of agents. In contrast, in this paper we allow a firm to choose not only the payment schemes of its employees but also the number of homogeneous workers it will employ. By giving the firm this additional instrument – namely, the number of employees –, we can analyze the trade-off between the extensive and intensive margins in a relational contracting setting with private information.

Naturally, when the firm and workers are sufficiently patient, the first-best action profile is attainable. For intermediate values of the discount factor, however, the lack of commitment constrains the total amount of bonuses the firm can promise, which implies inefficiencies and that at least one of the margins has to be distorted relative to first-best. The reduction in the total amount of bonuses may come by requiring less effort from employed workers or by reducing the number of employees. In particular, one thing that differentiates the two instruments is that the firm's cost is linear in the number of workers but convex in their effort. Interestingly, when second-best requires a downward distortion on the level of effort, this distortion can be paired with an upward distortion on the number of workers. Moreover, since the first-best level of effort does not vary with productivity, the dynamic observed when productivity falls is one in which some employees are laid off and the remaining ones start to exert more effort than before. Importantly, this increase happens due to a shift from an inefficient level of effort to an efficient level, and not the opposite.

On the other hand, the presence of private information creates an informational rent that tightens the firm's incentive compatibility constraints. In this case, the optimal self-enforcing relational plan (as we call a pair of contracts and the number of employees) presents history dependence, as private deviations cannot be punished. For intermediate values of the discount factor, in a high-productivity state, the firm distorts the effort exerted downwards relative to first-best and may distort the number of employees downwards or upwards, depending on the cost and production functions, similar to what we observe in the complete information case. Moreover, when productivity falls, the firm restores the first-best level of effort, thus increasing the effort of the remaining employees. On the other hand, with private information optimality requires a bigger layoff, with the number of employees falling below the low-productivity first-best level. But the firm starts to hire new employees in subsequent periods with the number of employees approaching the low-productivity first-best in the long run. This result suggests that the intensive margin is distorted only due to commitment issues, while the extensive margin is distorted also due to informational asymmetries.

Section 2 establishes the model and its primitives. Section 3 characterizes the first-best action profiles, as we call a pair of effort levels and a number of employees, for both productivity levels as a benchmark. Section 4 characterizes the optimal self-enforcing relational plan when information is complete. Section 5 characterizes the optimal self-enforcing relational plan when the firm's productivity is its private information. Section 6 presents concluding remarks and suggests further research and extensions of the framework in this paper.

Related Papers

Levin (2002) analyses the trade-offs between multilateral and bilateral relational contracts when information is complete, but adjustments may be costly due to communication problems. In his paper, the number of players is fixed and the necessity of layoffs is assumed. Garrett and Pavan (2012) consider a model of turnover with a stochastic environment, but in their model, the firm can commit to conditional payments. Andrews and Barron (2016) and Board (2011) consider models with multiple agents, but in which the principal chooses only one with whom to contract in each period. Kvaloy and Olsen (2006) and Mukherjee and Vasconcelos (2011) study models of relational contracting in teams. Fahn and Klein (2019), Li and Matouschek (2013), and Halac and Yared (2014) study models of relational contracts under asymmetric information and in a stochastic environment. With asymmetric information, optimal relational contracts are not stationary anymore due to informational rents. In contrast with the first two, Halac and Yared (2014) studies a model with persistent (but non-permanent) private shocks. We extend the literature by considering the case with permanent private shock but with an endogenous number of agents so as to study the extensive and intensive margins in a relational contracting setting.

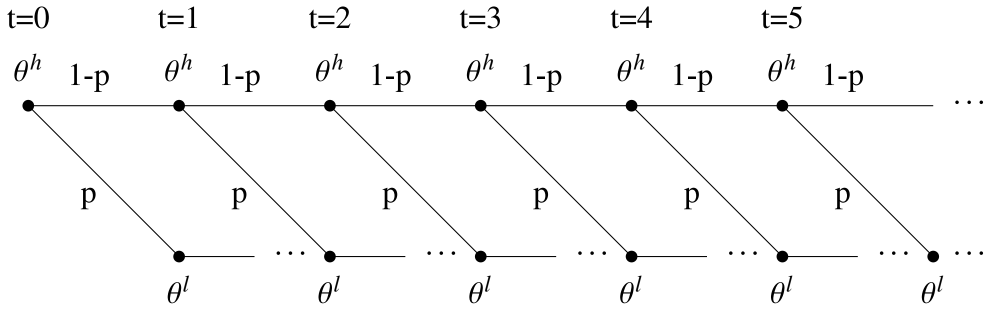
2 Model Setup

Consider the relationship between a firm (firm and manager will be used interchangeably from now on) and a continuum of homogeneous workers in a repeated game. The firm's production is a function of the number of workers $N \in [0, \infty)$ and the effort $e(i) \in [0, \overline{e}]$ that each worker $i \in [0, N]$ exert (where $\overline{e}$ is assumed to be sufficiently large). We assume that production (or revenue, with the product's price being normalized to 1) is given by $\theta\psi\left(\int_0^N e(i)di\right)$,[1] where ψ is a C^3 function with $\psi' > 0$, $\psi'' < 0$, $\psi(0) = 0$, and $\lim_{x\to\infty}\psi'(x) = 0$ and $\lim_{x\to 0}\psi'(x) = \infty$, and $\theta > 0$ is a parameter that represents the firm's productivity. We assume that effort and product are publicly observable but not verifiable by a court.

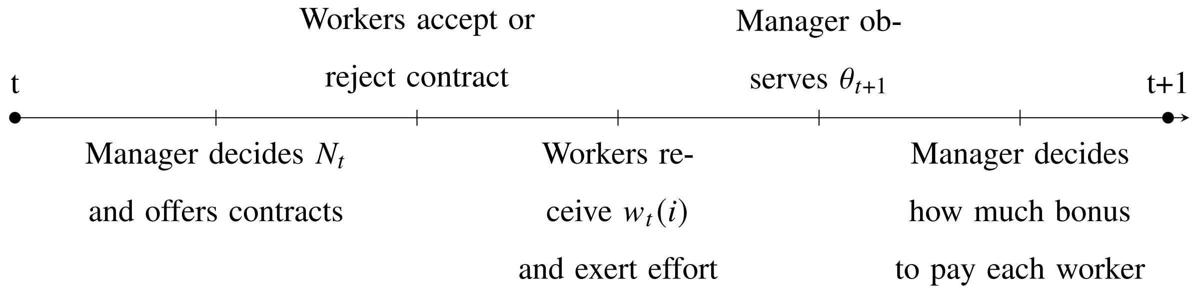
θ_t might change over time. We focus on the case in which $\theta_t \in \{\theta^l, \theta^h\}$, $\theta^l < \theta^h$, follows a simple Markov process that represents a permanent productivity shock: $\theta_0 = \theta^h$, $P(\theta_{t+1} = \theta^l|\theta_t = \theta^h) = p$, and $P(\theta_{t+1} = \theta^l|\theta_t = \theta^l) = 1$, that is, θ may decrease at some point in time and, from

[1] Since the workers are homogeneous, we assume that production is a function only of the total effort employed by the labor force, which is $\int_0^N e(i)di$. Of course, in general, production could depend on $e(\cdot)$ and N on a variety of ways through an operator $\psi(e, N)$.

then on, it stays low, as represented below. $\theta_0 = \theta^h$ and p are assumed to be public knowledge throughout the paper. In Sections 4 and 5 we explore the cases in which θ_t is publicly observable and privately observable by the manager, respectively.



The timing, represented below, is as follows: in the beginning of each period $t = 0, 1, \ldots,$ the manager decides how many workers to employ, N_t, and offers to each agent $i \in [0, N_t]$ a short-term contract $(w_t(i), b_t^h(i,e), b_t^l(i,e))$ that specifies a wage $w_t(i)$ paid before effort is exerted, and a bonus $b_t^k(i,e)$, paid after the worker chooses effort e and if $\theta_{t+1} = \theta^k$. Each worker $i \in [0, N_t]$, in turn, decides to accept or not the contract offered, which is represented by the indicator variable $d_t(i)$ that takes on the value 1 if the contract is accepted and 0 otherwise, and how much effort $e_t(i)$ to exert in the case the contract is accepted. At the end of period t, after effort $e_t(i)$ is exerted, but before the bonuses are paid, the manager observes the productivity θ_{t+1} of the next period. After that, she decides how much to pay each worker $b_t(i,e)$, then time $t+1$ starts.



We refer to the sequence of actions and contracts $\mathcal{P} = (N_t, e_t(\cdot), w_t(\cdot), b_t^h(\cdot), b_t^l(\cdot), b_t(\cdot))_{t \geq 0}$ as a relational plan[2], and say that a plan is self-enforcing if no one has incentive to deviate from it. It is known that dynamic games can have multiple different equilibria. We then focus on those that maximize the firm's ex-ante expected profits and refer to such relational plans as optimal self-enforcing relational plans. We also refer to the pair $(N_t, e_t(\cdot))_t$ as an action profile. A worker who exerts effort e_t incurs in cost $c(e_t)$, where c is a C^2 function with $c' > 0$,

[2]We use the term plan to differentiate the present setting from the standard one in which the number of players is fixed, although it carries the same meaning as the term relational contract.

$c'' > 0$, and $c(0) = 0$. We assume further that the firm is risk-neutral and that every agent has a quasi-linear utility function in money and a reservation utility equal to $\overline{u} > 0$.

We make the following assumptions regarding the relationship between the firm and its employees:

ASSUMPTION 1:

1.1 The firm incurs no layoff costs.

ASSUMPTION 2:

2.1 Whenever the firm lays off employees, that is, when $N_{t+1} < N_t$, dismissals are randomly assigned.

2.2 Whenever an employee is dismissed, he is never rehired in the future.

2.3 Whenever the firm hires new employees, that is, when $N_{t+1} > N_t$, all current ones are offered a new contract.

We believe Assumption 2 to be consistent with the model we propose, since the workers are homogeneous. As a whole, Assumption 2 implies that we can interpret each $i \leq N_t$ as a filled position in the firm instead of a label for a particular employee: two different agents can fill the job position i at different points in time (the difference will be made clearer in the next subsection when introducing strategies and payoffs). It also implies that we only need to keep track of the decisions and payoffs of a particular agent while he is employed. Regardless, we still refer to each i as an employee, as introduced before, and refer to i as an agent only when we want to make his identity explicit.

2.1 Strategies and payoffs

Let $h^{t-1} = \left(N_\tau, \left\{d_\tau(i), e_\tau(i), w_\tau(i), b^h_\tau(i), b^l_\tau(i), b_\tau(i)\right\}_{i\geq 0}\right)_{\tau=0}^{t-1}$ be a public history of actions taken and contracts offered (we omit the extra arguments of each function) from time $\tau = 0$ to $\tau = t - 1$, inclusive, and $\mathcal{H}^{t-1}$ be the set of all possible public histories of length t. h^{t-1} is assumed to be observed by everyone, including workers currently not employed. Analogously, let $h^{f,t-1} = \left(\theta_{\tau+1}, N_\tau, \left\{d_\tau(i), e_\tau(i), w_\tau(i), b^h_\tau(i), b^l_\tau(i), b_\tau(i)\right\}_{i\geq 0}\right)_{\tau=0}^{t-1}$ be the private history, which is observed only by the firm, and $\mathcal{H}^{f,t-1}$ be the set of all possible private histories of length t. Let $\theta^t = (\theta_\tau)_{\tau=0}^t$ be the history of productivities and Θ^t the set of all possible productivity histories of length $t+1$ that happen with positive probability. Moreover, let $\Theta^{\tau}(\theta^t)$ be the set of continuation histories of length τ that happen with positive probability given that θ^t has already happened, that is, if $\theta^{\tau} \in \Theta^{\tau}(\theta^t)$, then $(\theta^t, \theta_{t+1}, \ldots, \theta_{\tau}) \in \Theta^{t+\tau}$.

Formally, a pure strategy for the firm is a sequence of functions $\sigma^f = (N_t, w_t, b_t^h, b_t^l, b_t)_{t\geq 0}$ in which $N_t : \mathcal{H}^{f,t-1} \to \mathbb{R}_+$, $h^{f,t-1} \mapsto N_t(h^{f,t-1})$, $w_t : \mathbb{R}_+ \times \mathcal{H}^{f,t-1} \to \mathbb{R}$, $(i, h^{f,t-1}) \mapsto w_t(i, h^{f,t-1})$, $b_t^h : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R}_+$, $(i, h^{f,t-1}) \mapsto b_t^h(i, h^{f,t-1})$, $b_t^l : \mathbb{R}_+ \times \mathcal{H}^{f,t-1} \to \mathbb{R}_+$, $(i, h^{f,t-1}) \mapsto b_t^l(i, h^{f,t-1})$, and finally $b_t : \mathbb{R}_+ \times \mathcal{H}^{f,t-1} \times \mathbb{R}_+^3 \times \{0,1\} \times \mathbb{R}_+ \times \{\theta^h, \theta^l\} \to \mathbb{R}_+$, $(i, h^{f,t-1}, b_t^h, b_t^l, w_t, d_t, e_t, \theta_{t+1}) \mapsto b_t(i, h^{f,t-1}, b_t^h, b_t^l, w_t, d_t, e_t, \theta_{t+1})$ for all $t = 0, 1, \ldots$. As we have defined, we require b^h, b^l, and b to be non-negative but allow w to be negative. In other words, there is no limited liability in our model.

As we have briefly discussed when introducing Assumption 2, we only need to keep track of a particular agent's payoffs and strategies during the periods in which he is offered a contract. For an agent, the game starts when he is first offered a contract, and it ends either when he is laid off or when he does not accept the contract offered to him. Therefore, a pure strategy for a worker i is a sequence of functions $\sigma^w(i) = (d_t(i), e_t(i))_{t\geq 0}$ in which, conditional on having been offered a contract in t, $d_t(i, \cdot) : \mathcal{H}^{t-1} \times \mathbb{R} \times \mathbb{R}_+^2 \to \{0,1\}$, $(h^{t-1}, w_t, b_t^h, b_t^l) \mapsto d_t(i, h^{t-1}, w_t, b_t^h, b_t^l)$, and, conditional on having been offered a contract and having accepted, $e_t(i, \cdot) : \mathcal{H}^{t-1} \times \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}_+$, $(i, h^{t-1}, w_t, b_t^h, b_t^l) \mapsto e_t(i, h^{t-1}, w_t, b_t^h, b_t^l)$ for all $t = 0, 1, \ldots$. [3]

We focus on truth-telling pure strategies $\sigma = (\sigma^f, \sigma^w)$, that is, those in which the conditional payments are made accordingly to what have been promised in the contract, $b_t^k(i, \theta^t) = b_t(i, \theta^t, \theta^k)$ for $k = h, l$. As is standard, we assume that the firm and all workers discount time equally by the factor $\delta \in (0,1)$. Since we restrict attention to truth-telling pure strategies, σ_t is completely determined by the history θ^t. Under strategy profile σ, the firm's on-path expected profits at time t is given by

$$\pi_t(\theta^t) = \theta_t \psi \left(\int_0^{N_t(\theta^t)} e_t(i, \theta^t) di \right) - \int_0^{N_t(\theta^t)} \left(w_t(i, \theta^t) + \mathbb{E}_{\theta_t} \left[b_t(i, \theta^t) \right] \right) di$$

where $\mathbb{E}_{\theta_t} \left[b_t(i, \theta^t) \right] = \left(1 - P(\theta^l | \theta_t)\right) b_t^h(i, \theta^t) + P(\theta^l | \theta_t) b_t^l(i, \theta^t)$. Its continuation profits at time

[3] For continuation strategies to be correctly defined, when we have $N_{t+1} < N_t$, and since by Assumption 2 dismissals are random, the remaining N_{t+1} must be reassigned to positions $i \in [0, N_{t+1}]$ with the correct continuation strategy: if, for example, an agent who is represented by the index $N_{t+1} < i' < N_t$ at t, with continuation strategy given by $\sigma^w | h_t(i')$, remains employed in $t+1$, he must be reassigned uniquely to a position $i \in [0, N_{t+1}]$, for which we set $\sigma^w | h_t(i) = \sigma^w | h_t(i')$. Formally, we can reassign every $i' \in [0, N_t]$ uniquely to $i(i') = \dfrac{i'}{N_t} N_{t+1}$.

t and history θ^t is denoted by $\Pi_t(\theta^t)$ and given by $\Pi_t(\theta^t) = \mathbb{E}_{\theta_t}\left[\sum_{\tau=0}^{\infty}(1-\delta)\delta^{\tau+t}\pi_{\tau+t}(\theta^{\tau+t})\right]$, so its ex-ante expected profits is given by $\Pi(\sigma) = \Pi_0(\theta^h) = \mathbb{E}_{\theta^h}\left[\sum_{t=0}^{\infty}(1-\delta)\delta^t\pi_t(\theta^t)\right]$.[4] Similarly, the on-path expected utility at time t of a worker i (conditional on having been offered a contract) is given by

$$u_t(i,\theta^t) = d_t(i,\theta^t)\left[w_t(i,\theta^t) + \mathbb{E}_{\theta_t}\left[b_t(i,\theta^t)\right] - c(e_t(i,\theta^t))\right] + \left(1-d_t(i,\theta^t)\right)\overline{u}$$

with continuation utility given by $U_t(i,\theta^t) = \mathbb{E}_{\theta_t}\left[\sum_{\tau=0}^{\infty}(1-\delta)\delta^{\tau+t}u_{\tau+t}(\theta^{\tau+t})\right]$.[5] For future reference, let

$$\xi_t(\theta^t,\theta_{t+1}) = \max\left\{0, 1 - \frac{N_{t+1}(\theta^t,\theta_{t+1})}{N_t(\theta^t)}\right\}$$

be the probability of a worker i employed in period t be dismissed in $t+1$.[6]

3 Full commitment and first-best benchmark

As a benchmark, we look first at the full commitment, complete information optimal relational plan. Suppose that θ_t is publicly observable and that the manager can commit to a formal short-term contract. This means that the only constraints that the manager has to abide by are the workers' individual rationality constraints and that the first-best plan will entail decisions that depend only on each period's productivity. To incentivize effort e, the firm must pay a worker no less than $\overline{u}+c(e)$, and anything more than that can be improved upon by reducing it marginally. Moreover, since utility is quasi-linear, there is no loss in considering, for $k = h,l$, $w_t(i,\theta^k) = \overline{u}$ and any pair $\left(b_t^h(i,\theta^k), b_t^l(i,\theta^k)\right)$ such that $\mathbb{E}\left[b_t(i,\theta^k)\right] = \left(1-P(\theta^l|\theta^k)\right)b_t^h(i,\theta^k) + P(\theta^l|\theta^k)b_t^l(i,\theta^k) = c(e_t(i,\theta^k))$. This implies that all workers accept the contracts offered and that they receive their reservation utility $\overline{u}$. Additionally, the firm's profits employing N_t workers

[4] Since strategies are pure, the expectations are only taken over possible productivity histories. Multiplication by $(1-\delta)$ serves the purpose of normalizing the payoff.

[5] Formally, $U_t(i,\theta^t)$ refers to the continuation utility of the agent who is currently employed.

[6] For workers, since, by Assumption 2, dismissals are randomly assigned, expectations also weigh the possibility of being dismissed whenever there are layoffs. Then, conditional on having accepted the contract, we have

$$\begin{aligned} U_t(i,\theta^t) = {} & (1-\delta)\left[w_t(i,\theta^t) - c(e_t(i,\theta^t))\right] \\ & + \left(1-P(\theta^l|\theta_t)\right)\left[(1-\delta)b_t^h(i,\theta^t) + \left(1-\xi_t(\theta^t,\theta^h)\right)\delta U_{t+1}(i,\theta^t,\theta^h) + \xi_t(\theta^t,\theta^h)\delta\overline{u}\right] \\ & + P(\theta^l|\theta_t)\left[(1-\delta)b_t^l(i,\theta^t) + \left(1-\xi_t(\theta^t,\theta^l)\right)\delta U_{t+1}(i,\theta^t,\theta^l) + \xi_t(\theta^t,\theta^l)\delta\overline{u}\right] \end{aligned}$$

in period t is given by

$$\pi(\theta_t, N_t, e_t(\cdot)) = \theta_t \psi\left(\int_0^{N_t} e_t(i)di\right) - N_t\overline{u} - \int_0^{N_t} c(e_t(i))di$$

Let $(N^{fb,k}, e^{fb,k}(\cdot)) \in \max_{N,e(\cdot)} \pi(\theta^k, N, e(\cdot))$, $k = h, l$. From now on, we assume that the primitives of the model are such that $\max_{N,e(\cdot)} \pi(\theta^k, N, e(\cdot)) > 0$ for $k = h, l$ – that is, there are gains from trade. Since workers are identical and c is strictly convex, we have that a necessary condition for $e^{fb,k}(\cdot)$ to be a solution to the problem is $e^{fb,k}(i) = e^{fb,k}$ for all $i \in [0, N^{fb,k}]$. This implies that we can rewrite the maximization problems as

$$\max_{N,e} \theta^k \psi(Ne) - N\overline{u} - Nc(e) \tag{1}$$

The first-order conditions of the program 1 above are

$$[N]\ \theta^k \psi'\left(N^{fb,k} e^{fb,k}\right) e^{fb,k} = \overline{u} + c(e^{fb,k})$$
$$[e]\ \theta^k \psi'\left(N^{fb,k} e^{fb,k}\right) N^{fb,k} = c'(e^{fb,k}) N^{fb,k}$$

if $e^{fb,k}, N^{fb,k} > 0$. Diving $[e]$ by $N^{fb,k}$, multiplying by $e^{fb,k}$, and substituting $\theta^k \psi'\left(N^{fb,k} e^{fb,k}\right) e^{fb,k} = c'(e^{fb,k}) e^{fb,k}$ in $[N]$, we get

$$c'(e^{fb,k}) e^{fb,k} - c(e^{fb,k}) = \overline{u}$$

Since $\overline{u} > 0$ and c is strictly increasing and convex, there exists a unique $e^{fb,k} > 0$ that satisfies this equality. Given $e^{fb,k}$, we get that $N^{fb,k} = \frac{1}{e^{fb,k}} \psi'^{-1}\left(\frac{c'(e^{fb,k})}{\theta^k}\right) > 0$.[7]

As we have defined, in the first-best relational plan, whenever $\theta_t = \theta^k$, $k = h, l$, the firm would employ $N^{fb,k}$ workers and offer a contract with $w^{fb,k} = \overline{u}$ and $\mathbb{E}\left[b(i, \theta^k)\right] = c(e^{fb,k})$. Note that $e^{fb,k}$ is strictly increasing in $\overline{u}$, but does not depend on θ^k, and $N^{fb,k}$ is strictly decreasing on $\overline{u}$[8] and strictly increasing on θ^k. This implies $e^{fb,h} = e^{fb,l}$ and $N^{fb,h} > N^{fb,l}$. Intuitively, effort is chosen to equate marginal productivity to marginal cost. Since marginal cost does not change when a productivity shock occurs, it is more efficient to adapt to new conditions by hiring or laying off employees.

[7]The Hessian matrix of $\pi(\theta^k, N, e(\cdot))$ is negative definite at $(N^{fb,k}, e^{fb,k})$. The convexity of c, concavity of ψ, and $\overline{u} > 0$ also imply that it is also a global maximum.

[8]Suppose, for instance, that a minimum wage $\overline{w} > \overline{u}$ were to be required. Than the firm would substitute away from hiring and towards requiring higher levels of effort from its employees.

Let $\pi^{fb,k} = \theta^k \psi\left(N^{fb,k} e^{fb,k}\right) - N^{fb,k}\overline{u} - N^{fb,k} c(e^{fb,k})$. By the Envelope Theorem, $\pi^{fb,h} > \pi^{fb,l}$. If we let $\Pi^{fb,h}$ and $\Pi^{fb,l}$ be the firm's continuation profits under the first-best relational plan, then the ex-ante expected profits $\Pi^{fb,h}$ are given by the system

$$\begin{cases} \Pi^{fb,h} = (1-\delta)\pi^{fb,h} + \delta\left[(1-p)\Pi^{fb,h} + p\Pi^{fb,l}\right] \\ \Pi^{fb,l} = (1-\delta)\pi^{fb,l} + \delta\Pi^{fb,l} \end{cases}$$

which implies $\Pi^{fb,l} = \pi^{fb,l}$ and $\Pi^{fb,l} = \dfrac{1-\delta}{1-(1-p)\delta}\pi^{fb,h} + \dfrac{p\delta}{1-(1-p)\delta}\pi^{fb,l}$.

4 Relational Plans under Complete Information

Assume now that the firm cannot formally commit to the conditional payments in the contract, but the information is still complete. We will restrict attention to a particular class of equilibria in which any deviations from expected actions are followed by termination of the relationship between the firm and workers in all periods, which, as in Levin (2002), we call multilateral relational plans. We realize this is a rather strong restriction and aim to allow for a more flexible contractual structure in future research. One interpretation for these equilibria is that deviations from the firm, although affecting only a group of workers, might imply a change of behavior that could affect the firm's commitment to the payments of bonuses to all future workers, as if deviations had a reputational effect. A different but complementary interpretation concerns the behavior of unionized employees. In addition to simplicity, the choice of considering multilateral contracts is due to the fact that when plans are multilateral, punishments to deviations of the firm are larger, which implies that the firm can more strongly commit to conditional payments.

In this section, since the information is complete, the relevant equilibrium concept is Subgame Perfect Equilibrium, and we say that a relational plan is self-enforcing if no player has incentives to deviate from the strategies that generate it. By the One-shot Deviation Principle, and since we are considering contracts in which deviations are followed by reversion to the static (self-absorbing) equilibrium, the profile σ is an SPE if, and only if, there is no profitable one-shot deviation following a history in which no deviation has occurred before.[9] See that the most profitable deviation for a worker is to choose effort $e = 0$ since every deviation is punished by the termination of the relationship. Similarly, the most profitable deviation for the

[9] Since e is observed, there is no loss in considering $b^k(i,\theta^t)$ as the payment if $e = e(i,\theta^t)$ and $\theta_{t+1} = \theta^k$ and to put $b = 0$ for all other possible effort levels.

firm is to not pay any worker, that is, to set $b = 0$ (Of course, this is implied by our restriction to multilateral relational plans. If, for instance, the relationship is only dissolved among those directly affected by deviations, the firm could, in principle, decide to renege on the payments of only a fraction of its workers. In this case, the aforementioned deviation does not dominate all others necessarily). Therefore, the following conditions are necessary and sufficient for a plan generated by σ to be self-enforcing:

(IR^w) For all $t \geq 0$ and $\theta^t \in \Theta^t$ and all $i \leq N_t(\theta^t)$, $U_t(i,\theta^t) \geq \overline{u}$.

(IR^f) For all $t \geq 0$ and $\theta^t \in \Theta^t$, $\Pi_t(\theta^t) \geq 0$.

(IC^w) For all $t \geq 0$ and $\theta^t \in \Theta^t$ and all $i \leq N_t(\theta^t)$,

$$\begin{aligned}\delta\overline{u} \leq & -(1-\delta)c(e_t(i,\theta^t)) \\ & + \left(1 - P(\theta^l|\theta_t)\right)\left[(1-\delta)b_t^h(i,\theta^t) + \left(1-\xi_t(\theta^t,\theta^h)\right)\delta U_{t+1}(i,\theta^t,\theta^h) + \xi_t(\theta^t,\theta^h)\delta\overline{u}\right] \\ & + P(\theta^l|\theta_t)\left[(1-\delta)b_t^l(i,\theta^t) + \left(1-\xi_t(\theta^t,\theta^l)\right)\delta U_{t+1}(i,\theta^t,\theta^l) + \xi_t(\theta^t,\theta^l)\delta\overline{u}\right]\end{aligned}$$

$(IC^{f,h})$ For all $t \geq 0$ and $\theta^t \in \Theta^t$ such that $(\theta^t,\theta^h) \in \Theta^{t+1}$,

$$-(1-\delta)\int_0^{N_t(\theta^t)} b_t^h(i,\theta^t)di + \delta\Pi_{t+1}(\theta^t,\theta^h) \geq 0$$

$(IC^{f,l})$ For all $t \geq 0$ and $\theta^t \in \Theta^t$ such that $(\theta^t,\theta^l) \in \Theta^{t+1}$,

$$-(1-\delta)\int_0^{N_t(\theta^t)} b_t^l(i,\theta^t)di + \delta\Pi_{t+1}(\theta^t,\theta^l) \geq 0$$

Conditions (IR^w) and (IR^f) say that for both productivity levels, the equilibrium continuation payoffs are at least as large as the workers' and firm's outside options. Condition (IC^w) says that the sum of the expected bonuses payments and continuation utility in equilibrium must be larger than the utility the worker can get by deviating, $(1-\delta)c(e_t(i,\theta^t)) + \delta\overline{u}$. Conditions $(IC^{f,h})$ and $(IC^{f,l})$ say that the continuation profits for both possible next-period productivity levels must be larger than the profits the firm can get by reneging on the payments, $(1-\delta)\int_0^{N_t(\theta^t)} b_t^k(i,\theta^t)di$.

We look for plans $\mathcal{P}^\star = (N_t(\theta^t), e_t(i,\theta^t), w_t(i,\theta^t), b^h(i,\theta^t), b^l(i,\theta^t))_{t\geq 0, i\geq 0}$ that maximize the firm's ex-ante profits subject to the constraints (IR^w), (IR^f), (IC^w), $(IC^{f,h})$, and $(IC^{f,l})$, i.e.,

$$\begin{aligned}\mathcal{P}^{\star} \in \operatorname{argmax}_{\mathcal{P}} (1-\delta) &\left[\theta^h \psi \left(\int_0^{N_0(\theta^h)} e_0(i,\theta^h) di\right) - \int_0^{N_0(\theta^h)} w_0(i,\theta^h) di\right] \\ &+ (1-p)\left[-(1-\delta)\int_0^{N_0(\theta^h)} b_0^h(i,\theta^h) di + \delta \Pi_1(\theta^h,\theta^h)\right] \\ &+ p\left[-(1-\delta)\int_0^{N_0(\theta^h)} b_0^l(i,\theta^h) di + \delta \Pi_1(\theta^h,\theta^l)\right] \\ \text{s.t. } &(IR^w), (IR^f), (IC^w), (IC^{f,h}), (IC^{f,l})\end{aligned}$$

Lemma 1. *If there exists an optimal self-enforcing relational plan, then there exists another optimal self-enforcing relational plan* $\mathcal{P} = \left(N_t(\theta^t), e_t(i,\theta^t), w_t(i,\theta^t), b_t^h(i,\theta^t), b_t^l(i,\theta^t)\right)_{t\geq 0, i\geq 0}$ *for which the following properties hold for all* $t \geq 0$*, all* $\theta^t \in \Theta^t$*, and (almost) all* $i \leq N_t(\theta^t)$*:*

1. *Workers receive no rent:* $U_t(i,\theta^t) = \overline{u}$*;*

2. *The* (IC^w) *constraint holds with equality;*

3. $e_t(i,\theta^t)$ *is constant as a function of i, that is,* $e_t(i,\theta^t) = e_t(\theta^t)$*.*

Proof. See Appendix A. □

By Lemma 1, there is no loss in considering $b_t^k(i,\theta^t) = b_t^k(\theta^t)$ for all $i \leq N_t(\theta^t)$ and $k = h, l$. Moreover, it implies that $w_t(i,\theta^t) = \overline{u}$ for all $\theta^t \in \Theta^t$ and $i \leq N_t(\theta^t)$. The (IC^w) constraint (now, the same for all workers) becomes $\left(1 - P(\theta^l|\theta_t)\right) b_t^h(\theta^t) + P(\theta^l|\theta_t) b_t^l(\theta^t) = c(e_t(\theta^t))$ for all $t \geq 0$ and $\theta^t \in \Theta^t$. With these observations, we have that profits at time t and history θ^t can be written as

$$\pi_t(\theta^t) = \theta_t \psi\left(N_t(\theta^t) e_t(\theta^t)\right) - N_t(\theta^t)\left[\overline{u} + c(e_t(\theta^t))\right]$$

Additionally, the $(IC^{f,h})$ and $(IC^{f,l})$ constraints become $-(1-\delta)N_t(\theta^t)b_t^h(\theta^t) + \delta\Pi_{t+1}(\theta^t,\theta^h) \geq 0$ and $-(1-\delta)N_t(\theta^t)b_t^l(\theta^t) + \delta\Pi_{t+1}(\theta^t,\theta^l) \geq 0$ for all t and θ^t, respectively. In fact, we can combine these constraints into a dynamic enforcement constraint given by:

(DE^f) For all $t \geq 0$ and $\theta^t \in \Theta^t$,

$$\left(1 - P(\theta^l|\theta_t)\right)\Pi_{t+1}(\theta^t,\theta^h) + P(\theta^l|\theta_t)\Pi_{t+1}(\theta^t,\theta^l) \geq \frac{1-\delta}{\delta} N_t(\theta^t) c(e_t(\theta^t))$$

If $\mathcal{P}$ is an optimal self-enforcing relational plan, then (DE^f) clearly holds. On the other hand, if $\left(N_t(\theta^t), e_t(\theta^t)\right)_{t\geq 0}$ satisfies (DE^f) and maximizes ex-ante expected profits, then by putting

$$\begin{aligned}
w_t(\theta^t) &= \overline{u} \\
b_t^l(\theta^t) &= \min\left\{\frac{\delta}{(1-\delta)}\frac{\Pi_{t+1}(\theta^t,\theta^l)}{N_t(\theta^t)}, c(e_t(\theta^t))\right\} \\
b_t^h(\theta^t) &= \begin{cases} \dfrac{1}{\left(1-P(\theta^l|\theta_t)\right)}\left[c(e_t(\theta^t)) - P(\theta^l|\theta_t)b_t^l(\theta^t)\right] & \left(1-P(\theta^l|\theta_t)\right) \neq 0 \\ 0 & o/w \end{cases}
\end{aligned} \tag{2}$$

for all t and θ^t, we have that (IR^w) and (IC^w) hold with equality for every employee and every history, and $(IC^{f,h})$ and $(IC^{f,l})$ also hold by construction. Therefore, the relational plan $\mathcal{P} = \left(N_t(\theta^t), e_t(\theta^t), w_t(\theta^t), b_t^h(\theta^t), b_t^l(\theta^t)\right)_{t\geq 0}$ is self-enforcing and generates the same ex-ante expected profits. As a result, we only need to focus on finding the optimal action profile $\left(N_t(\theta^t), e_t(\theta^t)\right)_{t\geq 0}$ for which (DE^f) holds when we set contracts as in 2. Note that, although the distribution of bonuses between states depends on the future productivities, the total amount depends on the productivity, and hence on the profits, of the present period.

In Proposition 1, we show that, with complete information, we can restrict the search to stationary Markov plans with no loss of optimality, since all deviations are observed and punishments optimally occur only off the equilibrium path.

Proposition 1. *If there exists an optimal self-enforcing relational plan, then there exists another optimal self-enforcing relational plan* $\mathcal{P} = \left(N_t(\theta^t), e_t(i,\theta^t), w_t(i,\theta^t), b_t^h(i,\theta^t), b_t^l(i,\theta^t)\right)_{t\geq 0, i\geq 0}$ *in which* $\left(N_t(\theta^{t-1},\theta^k), e_t(\theta^{t-1},\theta^k)\right) = \left(N(\theta^k), e(\theta^k)\right)$, $k = h,l$, *for all* $t \geq 1$ *and* $\theta^{t-1} \in \Theta^{t-1}$, *that is,* $\mathcal{P}$ *is a stationary Markov relational plan.*

Proof. See Appendix A. □

Intuitively, as all deviations are observed and punished, the optimal relational plan does not need to involve distortions beyond those resulting from the lack of commitment. We focus on optimal relational plans that are also Markov for simplicity and alignment with the literature. Moreover, as we have discussed above, it is enough to consider only the (DE^f) constraint to find the optimal action profile. To ease notation burden, denote $N(\theta^k) = N^k$, $e(\theta^k) = e^k$, and $\pi^k = \theta^k\psi\left(N^k e^k\right) - N^k\overline{u} - N^k c(e^k)$, for $k = h,l$, and let $\Pi(\theta^k) = \Pi^k$ be the continuation profits for $k = h,l$. Note that if $k = l$, then $P(\theta^h|\theta^k) = 0$, so the $(DE^{f,l})$ constraint is simply $\Pi(\theta^l) \geq \dfrac{1-\delta}{\delta}N^l c(e^l)$; moreover, $\Pi(\theta^l) = \theta^l\psi\left(N^l e^l\right) - N^l\overline{u} - N^l c(e^l) = \pi^l$.

We therefore now define the optimal self-enforcing relational plan to be the one that satisfies the properties in Lemma 1 and Proposition 1, with contracts defined as in 2, which solves the program 3 below.

$$\max_{\left(N^k,e^k\right)_{k=h,l}} \quad \Pi^h = \frac{1-\delta}{1-(1-p)\delta}\pi^h + \frac{p\delta}{1-(1-p)\delta}\pi^l$$
$$\text{s.t.} \begin{cases} \dfrac{(1-p)(1-\delta)\delta}{1-(1-p)\delta}\pi^h + \dfrac{p\delta}{1-(1-p)\delta}\pi^l \geq (1-\delta)N^h c(e^h) & (DE^{f,h}) \\ \delta\pi^l \geq (1-\delta)N^l c(e^l) & (DE^{f,l}) \end{cases} \tag{3}$$

Proposition 2. *Let* $(N^h N^l, e^h, e^l)$ *be the action profile in an optimal self-enforcing relational plan. There exist* $p^\star \in [0,1)$ *and* $\delta^l_{ci}, \delta^{1,h}_{ci}(p) \in (0,1)$ *such that*

1. *If* $\delta > \max\left\{\delta^l_{ci}, \delta^{1,h}_{ci}(p)\right\}$, *then for both* $k = h,l$, $(N^k, e^k) = (N^{fb,k}, e^{fb,k})$;
2. *If* $p > p^\star$ *and* $\delta \in (\delta^l_{ci}, \delta^{1,h}_{ci}(p))$, *then* $(N^l, e^l) = (N^{fb,l}, e^{fb,l})$, $e^h < e^{fb,h}$, *and* $N^h < N^{fb,h}$ *(>) if* $f(e) := \theta^h \psi'(Ne)\left[e - \dfrac{c(e)}{c'(e)}\right]$ *is an increasing (decreasing) function of e for all N;*

   Moreover, if $p^\star > 0$, *then for* $p \in [0, p^\star)$, *there exists* $\delta^{2,h}_{ci}(p) \in (\delta^{1,h}_{ci}(p), \delta^l_{ci}) \neq \emptyset$ *such that*
3. *If* $p < p^\star$ *and* $\delta \in (\delta^{2,h}_{ci}(p), \delta^l_{ci})$, *then* $(N^h, e^h) = (N^{fb,h}, e^{fb,h})$, $e^l < e^{fb,l}$, *and* $N^l < N^{fb,l}$ *(>) if* $f(e) := \theta^l \psi'(Ne)\left[e - \dfrac{c(e)}{c'(e)}\right]$ *is an increasing (decreasing) function of e for all N;*
4. *If* $\delta < \min\left\{\delta^l_{ci}, \delta^{2,h}_{ci}(p)\right\}$, *then for both* $k = h,l$, $e^k < e^{fb,k}$ *and* $N^k < N^{fb,k}$ *(>) if* $f(e) := \theta^k \psi'(Ne)\left[e - \dfrac{c(e)}{c'(e)}\right]$ *is an increasing (decreasing) function of e for all N.*

Proof. See Appendix A. □

Note that $(N^k, e^k) = (N^{fb,k}, e^{fb,k})$, $k = h,l$, maximizes the objective function if no constraint is considered. For δ sufficiently large, this is the case. For intermediary values of δ, at least one of the two constraints will bind. First, see that the $(DE^{f,l})$ constraint only depends on (N^l, e^l). Now, consider the $(DE^{f,h})$ constraint when implementing first-best action profiles. The left-hand-side is a discounted average between profits in a high-productivity state and in a low-productivity state. As p grows bigger and the low-productivity state becomes more likely, the average approaches $\pi^{fb,l}$ and the constraint is tightened. On the other hand, if p decreases, the average approaches $\pi^{fb,h}$ and the constraint is loosened. We show that we can then establish the existence of a value $p^\star$ for which $(DE^{f,h})$ binds before $(DE^{f,l})$ if $p > p^\star$, in the sense that the interval $[\delta_{ci}^{1,h}(p), 1)$ of δ for which $(DE^{f,h})$ holds when implementing the first-best action profiles is strictly smaller than the interval $[\delta_{ci}^{l}, 1)$ of δ for which $(DE^{f,l})$ holds. If $p^\star > 0$, then for $p < p^\star$ and $\delta \in (\delta_{ci}^{1,h}(p), \delta_{ci}^{l})$, the $(DE^{f,l})$ constraint binds first and $(N^l, e^l) = (N^{fb,l}, e^{fb,l})$ is no longer implementable. In this case, $(DE^{f,l})$ will bind at the optimum, so $e^l < e^{fb,l}$ and $N^l \lesseqgtr N^{fb,l}$. But since $\pi^l < \pi^{fb,l}$ in this case, we can only guarantee that $(N^{fb,h}, e^{fb,h})$ is implementable for values of δ close to δ^l, that is $\delta \in (\delta_{ci}^{2,h}(p), \delta_{ci}^{l})$, with $\delta_{ci}^{2,h}(p) > \delta_{ci}^{1,h}(p)$.

Proposition 2 states that whenever one of the (DE^f) constraints binds, the respective effort level is distorted downwards but the number of workers may even be distorted upwards depending on the curvatures of the production and cost functions. The lack of commitment requires a reduction in total cost. Intuitively, since cost is convex in effort, the firm optimally decreases e, although this reduction can be compensate by hiring employees in excess relative to first-best, since an increase in N does not tighten individual incentive compatibility constraints.

Interestingly, since $e^{fb,h} = e^{fb,l}$, for $p > p^\star$ and $\delta \in (\delta_{ci}^{l}, \delta_{ci}^{1,h}(p))$, as the firm becomes less productive, that is in the transition from a period in which $\theta = \theta^h$ to a period in which $\theta = \theta^l$, we would observe layoffs (as $N^h > N^l$), but an increase in the effort exerted by the remaining employees (as $e^h < e^{fb,h} = e^{fb,l} = e^l$). Note, however, that the increase in effort happens because effort was inefficiently low with high productivity, not because it becomes inefficiently high when productivity falls. Figure 1 and Figure 2 present the dynamics of the action profiles in an optimal self-enforcing relational plan for $\delta \in (\delta_{ci}^{l}, \delta_{ci}^{1,h}(p))$ and $p > p^\star$, that is, for when $(DE^{f,h})$ binds first, for examples in which $N^h < N^{fb,h}$ and $N^h > N^{fb,h}$, respectively. The reason for our focus in this set of parameters is to be able to compare the optimal self-enforcing plan in both the complete and private information cases.

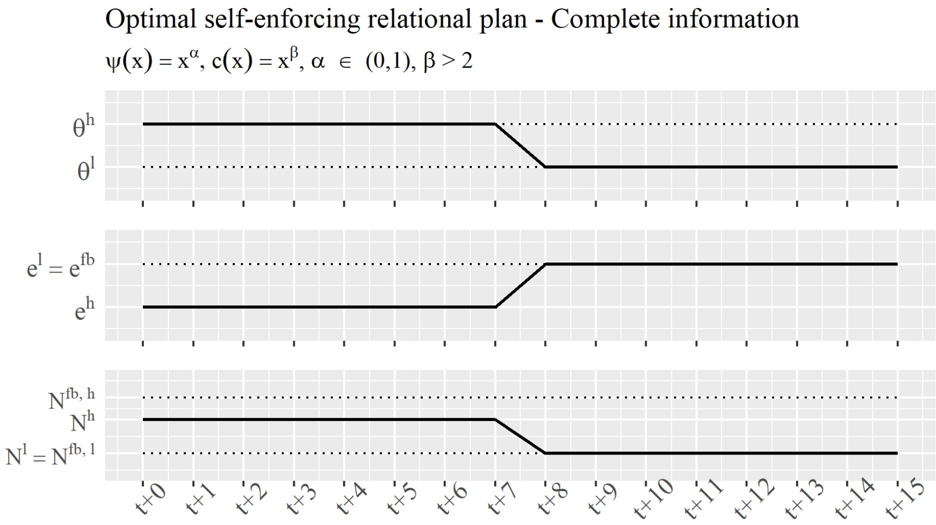


Figure 1: $\delta \in (\delta^{l}_{ci}, \delta^{1,h}_{ci}(p))$. Example with $N^{h} < N^{fb,h}$

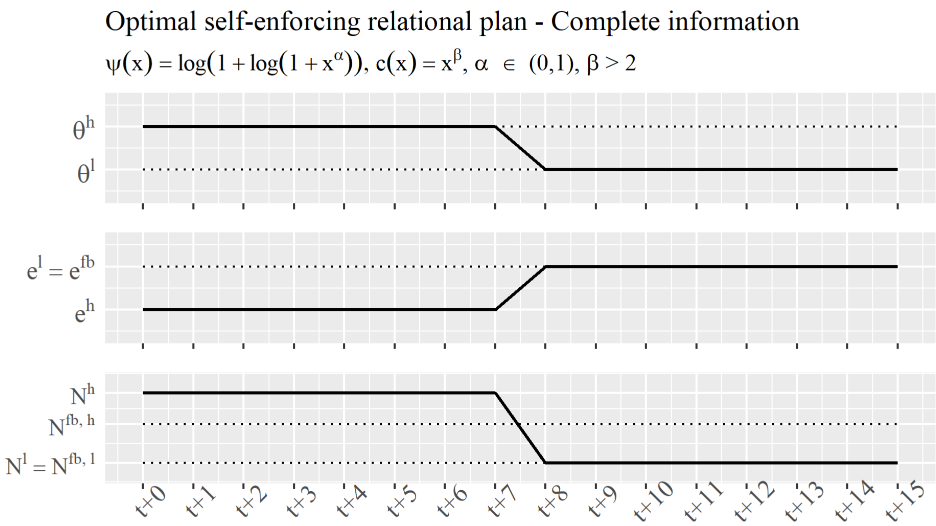


Figure 2: $\delta \in (\delta^{l}_{ci}, \delta^{1,h}_{ci}(p))$. Example with $N^{h} > N^{fb,h}$

5 Relational Plans under Private Information

In this section, we assume that θ is privately observed by the manager. The equilibrium concept used is Perfect Public Bayesian Equilibrium (PPBE), and, as mentioned before, we focus on pure public strategies.[10] In addition to the individual rationality constraints, (IR^{w}) and (IR^{f}), and the incentive compatibility constraints that prevent publicly observable deviations, (IC^{w}) and (IC^{f}), it must be in the manager's best interest to follow through her promises of payments

[10]The restriction to public strategies is a common one in the literature of relational contracts with private information.

in a truthful manner, that is, to pay $b_t(\theta^t, \theta^k) = b_t^k(\theta^t)$ whenever $\theta_{t+1} = \theta^k$.[11] With θ_t following the Markov process specified earlier, once the manager claims that the next period's productivity is low (by paying the bonus specified for θ^l), she effectively commits to play accordingly to $\theta_t = \theta^l$ in every period from then on: announcing a high type after having announced a low type at a prior period is a deviation that gives the firm a continuation profit of zero. If, following a history in which no public deviation had occurred and $\theta_{t+1} = \theta^h$ had been privately observed, the manager decided to privately deviate and claim that $\theta_{t+1} = \theta^l$ instead, her continuation payoff would be given by

$$\begin{aligned}
&\Pi_{t+1}^{dl}(\theta^t, \theta^h) \\
&\quad := (1-\delta)\left[\theta^h \psi\left(\int_0^{N_{t+1}(\theta^t,\theta^l)} e_{t+1}(i,\theta^t,\theta^l)di\right) - \int_0^{N_{t+1}(\theta^t,\theta^l)} w_{t+1}(i,\theta^t,\theta^l)di \right. \\
&\qquad \left. - \int_0^{N_{t+1}(\theta^t,\theta^l)} b_{t+1}^l(i,\theta^t,\theta^l)di\right] \\
&\quad + (1-p)\delta\Pi_{t+2}^{dl}(\theta^t,\theta^h,\theta^h) + p\delta\Pi_{t+2}(\theta^t,\theta^l,\theta^l)
\end{aligned}$$

where $\Pi_{t+2}(\theta^t, \theta^l, \theta^l)$ is the on-path continuation profits following $(\theta^t, \theta^l, \theta^l)$.[12] That is to say, the manager's best course of action after the deviation is to act as if the productivity were indeed low. Note that we can rewrite $\Pi_{t+1}^{dl}(\theta^t, \theta^h)$ as

$$\begin{aligned}
\Pi_{t+1}^{dl}(\theta^t, \theta^h) &= \Pi_{t+1}(\theta^h, \theta^l) \\
&+ (1-\delta)(\theta^h - \theta^l)\sum_{\tau=0}^{\infty}((1-p)\delta)^\tau \psi\left(\int_0^{N_{t+1+\tau}(\theta^t,\theta^l,\ldots,\theta^l)} e_{\tau+1+\tau}(i,\theta^t,\theta^l,\ldots,\theta^l)di\right)
\end{aligned}$$

The second term in the RHS is strictly positive and represents the information rent of a high-type firm.

Analogously, if $\theta_{t+1} = \theta^l$ had been privately observed for the first time and the manager

[11] Payments $\left(b_\tau(\theta^\tau, \theta^k)\right)_{\tau=0}^t$ can be seen as a history of announcements. Indeed, by paying $b_\tau(\theta^\tau, \theta^k)$ the firm's is, in essence, communicating with the workers about the next period's productivity. If $b_\tau^h(\theta^\tau) = b_\tau^l(\theta^\tau)$ for some θ^τ, we assume that the firm also announces next period's type at the time of payments in order to allow coordination.

[12] The superscript dl means "deviation to low".

decided to claim that $\theta_{t+1} = \theta^h$ instead, her continuation payoff would be given by

$$
\begin{aligned}
&\Pi^{dh}_{t+1}(\theta^t, \theta^l) \\
&:= (1-\delta)\left[\theta^l \psi\left(\int_0^{N_{t+1}(\theta^t,\theta^h)} e_{t+1}(i,\theta^t,\theta^h)di\right) - \int_0^{N_{t+1}(\theta^t,\theta^h)} w_{t+1}(i,\theta^t,\theta^h)di\right] \\
&+ \max\left\{-(1-\delta)\int_0^{N_{t+1}(\theta^t,\theta^h)} b^h_{t+1}(i,\theta^t,\theta^h)di + \delta\Pi^{dh}_{t+2}(\theta^t,\theta^l,\theta^l),\right. \\
&\qquad\left. -(1-\delta)\int_0^{N_{t+1}(\theta^t,\theta^h)} b^l_{t+1}(i,\theta^t,\theta^h)di + \delta\Pi_{t+2}(\theta^t,\theta^h,\theta^l)\right\}
\end{aligned}
$$

that is, once θ^l is privately observed by the manager, she knows with certainty the productivity of all future periods and, thus, she can decide when to make the information public by claiming θ^l.

In any self-enforcing relational plan, we also require the following truth-telling constraints to hold:

(TT^h) For all $t \geq 0$ and $\theta^t \in \Theta^t$ such that $(\theta^t, \theta^h) \in \Theta^{t+1}$,

$$
-(1-\delta)\int_0^{N_t(\theta^t)} b^h_t(i,\theta^t)di + \delta\Pi_{t+1}(\theta^t,\theta^h) \geq -(1-\delta)\int_0^{N_t(\theta^t)} b^l_t(i,\theta^t)di + \delta\Pi^{dl}_{t+1}(\theta^t,\theta^h)
$$

(TT^l) For all $t \geq 0$ and $\theta^t \in \Theta^t$ such that $(\theta^t, \theta^l) \in \Theta^{t+1}$,

$$
-(1-\delta)\int_0^{N_t(\theta^t)} b^l_t(i,\theta^t)di + \delta\Pi_{t+1}(\theta^t,\theta^l) \geq -(1-\delta)\int_0^{N_t(\theta^t)} b^h_t(i,\theta^t)di + \delta\Pi^{dh}_{t+1}(\theta^t,\theta^l)
$$

The optimal self-enforcing relational plan $\mathcal{P}^\star$ maximizes the firm's ex-ante expected profits subject to the constraints (IR^w), (IR^f), (IC^w), (IC^f), (TT^h), and (TT^l).

Firstly, as $b^k_t(i,\theta^t) \geq 0$ for all $\theta^t \in \Theta^t$, the $(IC^{f,h})$ and $(IC^{f,l})$ constraints imply the (IR^f) constraint. Lemma 2 also states that we also do not need to consider constraint $(IC^{f,h})$.

Lemma 2. *Constraints $(IC^{f,l})$ and (TT^h) imply constraint $(IC^{f,h})$.*

Proof. As $\theta^h > \theta^l$ and

$$\begin{aligned}\Pi^{dl}_{t+1}(\theta^t,\theta^h) &= \Pi_{t+1}(\theta^h,\theta^l)\\ &\quad + (1-\delta)(\theta^h-\theta^l)\sum_{\tau=0}^{\infty}((1-p)\delta)^\tau \psi\left(\int_0^{N_{t+1+\tau}(\theta^t,\theta^l,\ldots,\theta^l)} e_{\tau+1+\tau}(i,\theta^t,\theta^l,\ldots,\theta^l)di\right)\\ &> \Pi_{t+1}(\theta^h,\theta^l)\end{aligned}$$

we have

$$\begin{aligned}-(1-\delta)\int_0^{N_t(\theta^t)} b^h_t(i,\theta^t)di + \delta\Pi_{t+1}(\theta^t,\theta^h) &\geq -(1-\delta)\int_0^{N_t(\theta^t)} b^l_t(i,\theta^t)di + \delta\Pi^{dl}_{t+1}(\theta^t,\theta^h)\\ &\geq -(1-\delta)\int_0^{N_t(\theta^t)} b^l_t(i,\theta^t)di + \delta\Pi_{t+1}(\theta^t,\theta^l)\\ &\geq 0\end{aligned}$$

□

Therefore, we only need to consider constraints (IR^w), (IC^w), $(IC^{f,l})$, (TT^h), and (TT^l).

Similarly to what we did in the case with complete information, we state Lemma 3, with the proof following Lemma 1's proof almost unchanged.

Lemma 3. *If there exists an optimal self-enforcing relational plan, then there exists another optimal self-enforcing relational plan* $\mathcal{P} = \left(N_t(\theta^t), e_t(i,\theta^t), w_t(i,\theta^t), b^h_t(i,\theta^t), b^l_t(i,\theta^t)\right)_{t\geq 0, i\geq 0}$ *for which the following properties hold for all* $t \geq 0$, *all* $\theta^t \in \Theta^t$, *and (almost) all* $i \leq N_t(\theta^t)$*:*

1. *Workers receive no rent:* $U_t(i,\theta^t) = \overline{u}$;
2. $e_t(i,\theta^t)$ *is constant as a function of i, that is,* $e_t(i,\theta^t) = e_t(\theta^t)$.

Proof. See Appendix B. □

By Lemma 3.1, we can consider relational plans in which $u_t(i,\theta^t) = \overline{u}$ for all $\theta^t \in \Theta^t$, so

$$w_t(i,\theta^t) + \left(1 - P(\theta^l|\theta_{t_1})\right) b^h_t(i,\theta^t) + P(\theta^l|\theta_{t_1}) b^l_t(i,\theta^t) = c(e_t(i,\theta^t)) + \overline{u}$$

This means that in an optimal self-enforcing relational plan, it is not necessary to give workers inter-temporal incentives. Moreover, as $e_t(i,\theta^t) = e_t(\theta^t)$ for all $i \leq N_t(\theta^t)$, there is no loss of optimality in considering a uniform compensation scheme, that is, one in which $b_t^k(i,\theta^t) = b_t^k(\theta^t)$, $k = h,l$, and $w_t(i,\theta^t) = w_t(\theta^t)$ for all $i \leq N_t(\theta^t)$. Hence, each period's profits can be written as

$$\pi_t(\theta^t) = \theta_t\psi\left(N_t(\theta^t)e_t(\theta^t)\right) - N_t(\theta^t)\left[\overline{u} + c(e_t(\theta^t))\right]$$

In what follows, we derive some properties of the optimal self-enforcing relational plan of a relaxed problem in which the (TT^l) constraint is not considered, and then show at the end that this plan satisfies the (TT^l) constraint. However, *we keep the same notation throughout* and continue to call a relational plan that satisfies all other constraints besides (TT^l) a self-enforcing relational plan.

Let $n(\theta^t) : \bigcup_{t=1}^{\infty} \Theta^t \mapsto \mathbb{N} \cup \{0\}$ denote the number of periods since the last realization of θ^l in θ^t, that is, $n(\theta^t, \theta^h) = 0$ and $n(\theta^t, \theta^l) = n(\theta^t) + 1$ for all $\theta^t \in \Theta^t$.

Proposition 3. *If there exists an optimal self-enforcing relational plan, then there exists another optimal self-enforcing relational plan* $\mathcal{P} = \left(N_t(\theta^t), e_t(i,\theta^t), w_t(i,\theta^t), b_t^h(i,\theta^t), b_t^l(i,\theta^t)\right)_{t\geq 0, i\geq 0}$ *in which* $\left(N_t(\theta^t), e_t(\theta^t)\right) = \left(N_{t'}(\theta^{t'}), e_{t'}(\theta^{t'})\right)$ *for all* $\theta^t \in \Theta^t$ *and* $\theta^{t'} \in \Theta^{t'}$ *such that* $n(\theta^t) = n(\theta^{t'})$.

Proof. See Appendix B. □

With asymmetric information, private deviations cannot be punished. Moreover, the shock persistence implies that the continuation game after a private deviation differs from the original one, since the workers do not have complete information about the firm's expected payoffs. These characteristics prevent the type of stationarity observed in the case with complete information, as incentive compatibility now requires distortions on the on-path optimal relational plan to account for the informational rent that a high-productivity firm has.

Proposition 3 implies that we can focus on relational plans in which the action profiles (and contracts) only depend on the distance to the last announcement of a high-productivity state. Therefore, we can let $\left(N^h, e^h, w^h, b^{h,h}, b^{h,l}\right)$ be the action profile and contract offered when $\theta_t = \theta^h$ and $\left(N^l(n), e^l(n), w^l(n), b^l(n)\right)$, $n \in \mathbb{N}$, the action profile and contract offered when $\theta_t = \theta^l$ and $n(\theta^t) = n$. Moreover, let π^h and $\pi^l(n)$ be the respective current period profits, and

$$\begin{aligned}\Pi^l(n) &= (1-\delta)\pi^l(n) + \delta\Pi^l(n+1) \\ &= (1-\delta)\sum_{i=n}^{\infty} \delta^{i-n}\pi^l(i)\end{aligned}$$

and

$$\Pi^h = (1-\delta)\pi^h + (1-p)\delta\Pi^h + p\delta\Pi^l(1)$$
$$= \frac{(1-\delta)}{1-(1-p)\delta}\pi^h + \frac{p(1-\delta)}{1-(1-p)\delta}\sum_{n=1}^{\infty}\delta^n\pi^l(n)$$

the on-path continuation profits.

We can also write

$$\begin{aligned}\Pi^{dl}(n) = &(1-\delta)\left[\theta^h\psi\left(N^l(n)e^l(n)\right) - N^l(n)w^l(n) - N^l(n)b^l(n)\right]\\ &+(1-p)\delta\Pi^{dl}(n+1) + p\delta\Pi^l(n+1)\\ &= \Pi^l(n) + (1-\delta)(\theta^h-\theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^l(n+i)e^l(n+i)\right)\end{aligned}$$

for $n \in \mathbb{N}$.

The relaxed program can then be written as

$$\max_{\mathcal{P}} \Pi^h = \frac{(1-\delta)}{1-(1-p)\delta}\pi^h + \frac{p(1-\delta)}{1-(1-p)\delta}\sum_{n=1}^{\infty}\delta^i\pi^l(n)$$

$$\text{s.t.}\begin{cases}(1-p)b^{h,h} + pb^{h,l} \geq c(e^h) & (IC^{w,h})\\ b^l(n) \geq c(e^l(n)) \text{ for all } n \in \mathbb{N} & (IC^{w,l})\\ -(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) \geq 0 & (IC_h^{f,l})\\ -(1-\delta)N^l(n)b^l(n) + \delta\Pi^l(n+1) \geq 0 \text{ for all } n \in \mathbb{N} & (IC_l^{f,l})\\ -(1-\delta)N^h b^{h,h} + \delta\Pi^h \geq -(1-\delta)N^h b^{h,l} + \delta\Pi^{dl}(1) & (TT^h)\end{cases}$$

Lemma 4. *If there exists an optimal self-enforcing relational plan, then there exists another optimal self-enforcing relational plan* $\mathcal{P} = \left(N^h, e^h, w^h, b^{h,h}, b^{h,l}, N^l(n), e^l(n), w^l(n), b^l(n)\right)_{n\in\mathbb{N}}$ *in which* $b^{h,h} \geq b^{h,l}$.

Proof. See Appendix B. □

Lemma 4 also implies that $\Pi^h \geq \Pi^{dl}(1) \geq \Pi^l(1)$.

Lemma 5. *If there exists an optimal self-enforcing relational plan, then there exists another optimal self-enforcing relational plan* $\mathcal{P} = \left(N^h, e^h, w^h, b^{h,h}, b^{h,l}, N^l(n), e^l(n), w^l(n), b^l(n)\right)_{n\in\mathbb{N}}$

in which the (IC^w) constraint binds.

Proof. See Appendix B. □

As an implication of Lemma 3 and Lemma 5, we may set $w^h = w^l(n) = \overline{u}$ for all $n \in \mathbb{N}$. Lemma 5 also implies that we can write constraint $(IC_l^{f,l})$ as $-(1-\delta)N^l(n)c(e^l(n)) + \delta\Pi^l(n+1) \geq 0$ for all $n \in \mathbb{N}$. Moreover, if the action profile $\left(N^h, e^h\right)$ is part of an optimal self-enforcing relational plan and, thus, satisfies the $(IC^{f,l})$ and (TT^h) constraints, then it also satisfies the following $(DE^{f,h})$ constraint:

$(DE^{f,h})$

$$
\begin{aligned}
& -(1-\delta)N^h c(e^h) + \delta(1-p)\Pi^h + \delta p \Pi^l(1) \\
& \geq (1-\delta)(\theta^h - \theta^l) \sum_{n=1}^{\infty} ((1-p)\delta)^n \psi\left(N^l(n)e^l(n)\right)
\end{aligned}
$$

The $(DE^{f,h})$ constraint is obtained by multiplying the (TT^h) constraint by $(1-p)$ and summing $(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) \geq 0$ to it. In fact, the converse is also true: if the action profile $\left(N^h, N^l(n), e^h, e^l(n)\right)_{n\in\mathbb{N}}$ maximizes the ex-ante expected profits subject to the $(DE^{f,h})$ constraint, then there exists a self-enforcing relational plan that generates the same ex-ante expected profits and is, therefore, optimal. By letting

$$
\begin{aligned}
w^h &= \overline{u} \\
w^l(n) &= \overline{u} \\
b^l(n) &= c(e^l(n))
\end{aligned}
\tag{4}
$$

for all $n \in \mathbb{N}$, we only need to further specify $b^{h,h}$ and $b^{h,l}$. This can be done by setting

$$
\begin{aligned}
b^{h,l} &= \min\left\{\frac{\delta}{(1-\delta)}\frac{\Pi^l(1)}{N^h}, c(e^h)\right\} \\
b^{h,h} &= \frac{1}{(1-p)}\left[c(e^h) - p b^{h,l}\right]
\end{aligned}
\tag{5}
$$

By construction, $(1-p)b^{h,h} + p b^{h,l} = c(e^h)$, $b^{h,h} \geq b^{h,l}$, and $-(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) \geq 0$, so $(IC^{w,h})$ and $(IC_h^{f,l})$ hold. Furthermore, to see that the (TT^h) constraint holds, note that if $b^{h,l} = b^{h,h} = c(e^h) \leq \frac{\delta}{(1-\delta)}\frac{\Pi^l(1)}{N^h}$, $\Pi^h \geq \Pi^{dl}(1)$ implies the (TT^h) constraint. If $b^{h,l} = \frac{\delta}{(1-\delta)}\frac{\Pi^l(1)}{N^h}$, then by summing $-(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) = 0$ on both sides of $(DE^{f,h})$ and dividing it by

$(1-p)$ gives us the (TT^h) constraint. Therefore, in the relaxed problem, we can combine $(IC_h^{f,l})$ and (TT^h) into the constraint $(DE^{f,h})$ while setting $b^{h,h}$ and $b^{h,l}$ as in 5, which implies that we only need to consider the constraints $(IC_l^{f,l})$ and $(DE^{f,h})$ while choosing (N^h, e^h) and $(N^l(n), e^l(n))_{n\in\mathbb{N}}$ to maximize the firm's ex-ante expected profits. We therefore now define the optimal self-enforcing relational plan to be the one that satisfies the properties in Lemmas 3, 4, and 5, and in Proposition 3, with contracts defined as in 4 and 5, which solves the program 6 below

$$\max_{(N^k,e^k)_{k=h,l}} \Pi^h = \frac{(1-\delta)}{1-(1-p)\delta}\pi^h + \frac{p(1-\delta)}{1-(1-p)\delta}\sum_{n=1}^{\infty}\delta^n\pi^l(n)$$

$$\text{s.t.}\begin{cases} \dfrac{(1-p)(1-\delta)\delta}{1-(1-p)\delta}\pi^h + \dfrac{p(1-\delta)}{1-(1-p)\delta}\sum_{n=1}^{\infty}\delta^n\pi^l(n) \\ \geq (1-\delta)N^h c(e^h) + (1-\delta)(\theta^h-\theta^l)\sum_{n=1}^{\infty}((1-p)\delta)^n\psi\left(N^l(n)e^l(n)\right) & (DE^{f,h}) \\ (1-\delta)\sum_{i=n+1}^{\infty}\delta^{i-n}\pi^l(i) \geq (1-\delta)N^l(n)c(e^l(n)) \text{ for all } n\in\mathbb{N} & (IC_l^{f,l}) \end{cases} \tag{6}$$

We proceed to prove Proposition 4, which characterizes the solution to the relaxed problem for the set of parameters in which the $(DE^{f,h})$ constraint binds while the $(IC_l^{f,l})$ constraints do not. In the proof of Proposition 4, we show that this plan indeed also satisfies the (TT^l) constraint and, consequently, is the optimal self-enforcing relational plan.

Proposition 4. *Let* $\left(N^h, N^l(n), e^h, e^l(n)\right)_{n\in\mathbb{N}}$ *be the action profile in an optimal self-enforcing relational plan. There exist* $p^\star \in [0,1)$, *and* $\delta_{pi}^l = \delta_{ci}^l$, $\delta_{pi}^h(p) \in (0,1)$ *such that for* $p > p^\star$ *and* $\delta \in (\delta_{pi}^l, \delta_{pi}^h(p))$, *we have*

1. $e^h < e^{fb,h}$; $N^h < N^{fb,h}$ $(>)$ *if* $f(e) := \theta^h\psi'(Ne)\left[e - \dfrac{c(e)}{c'(e)}\right]$ *is an increasing (decreasing) function of* e *for all* N.

2. $N^l(n) < N^l(n+1) < N^{fb,l}$ *and* $e^l(n) = e^{fb,l}$ *for all* $n\in\mathbb{N}$.

3. $N^l(n) \xrightarrow{n} N^{fb,l}$.

Proof. See Appendix B. □

The private information tightens the firm's $(DE^{f,h})$ constraint when compared to the complete information case, as a private deviation to low is more profitable than reneging on the conditional payments. Note that the information rent is increasing with the level of production in low states, so optimality requires distortions on the action profile played whenever productivity is low. Since periods further in time are discounted at a geometric rate of $\delta(1-p)$, these distortions are greater when the low type is closer to the last announcement of a high type and they disappears in the long run. Intuitively, as the probability of productivity turning low is positive, an announcement of a low type becomes more plausible with time. Interestingly though, only the number of employees is distorted in a low productivity state. This result indicates that, in this setting, the firm distorts the effort level only due to commitment issues, but the number of employees also due to informational asymmetries.

As the firm becomes less productive, we would observe an initial layoff and an increase in the level of effort of the remaining employees. However, in contrast to the case with complete information, the initial layoff is bigger and the firm rehires new employees in subsequent periods. Figure 3 and 4 present the dynamics of the action profiles in an optimal self-enforcing relational plan for $\delta \in (\delta^l_{pi}, \delta^h_{pi}(p))$ and $p > p^\star$, that is, for when $(DE^{f,h})$ binds first, for examples in which $N^h < N^{fb,h}$ and $N^h > N^{fb,h}$, respectively

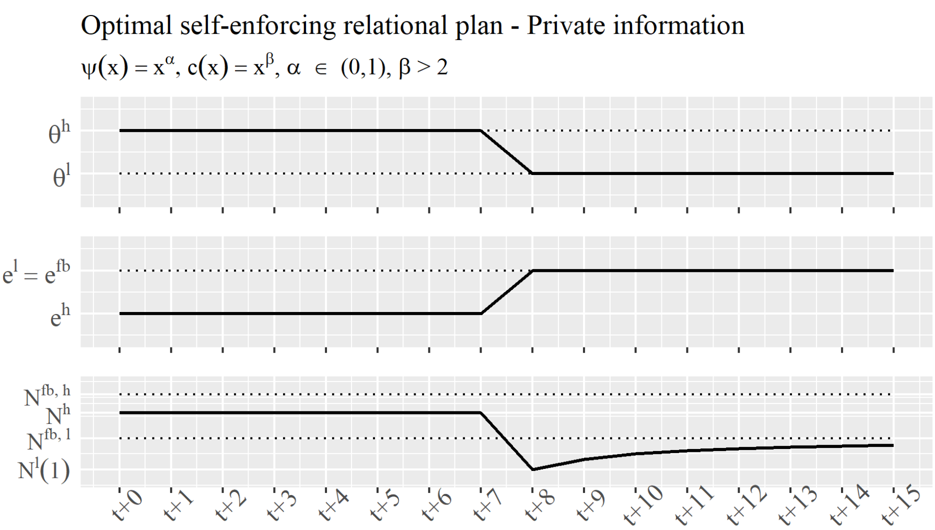


Figure 3: $\delta \in (\delta^l_{pi}, \delta^h_{pi}(p))$. Example with $N^h < N^{fb,h}$

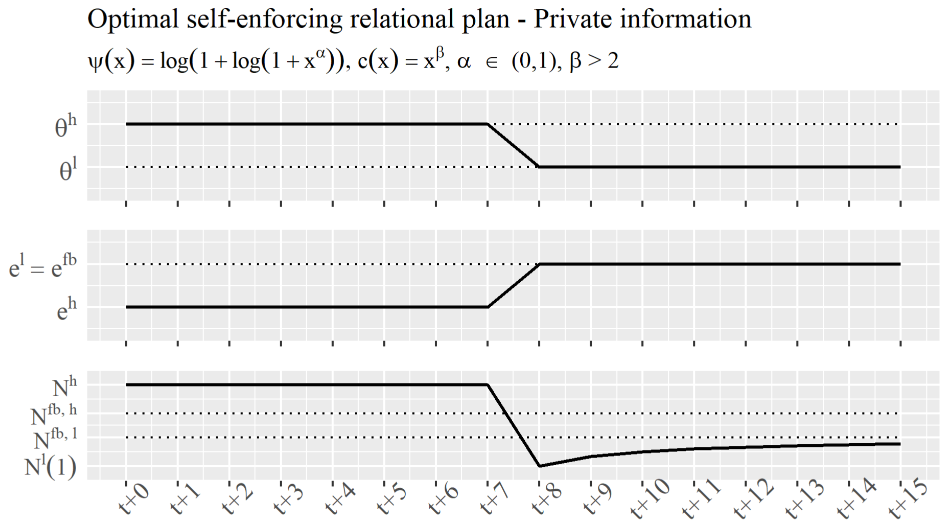


Figure 4: $\delta \in (\delta^l_{pi}, \delta^h_{pi}(p))$. Example with $N^h > N^{fb,h}$

6 Conclusion

This thesis has studied the extensive and intensive margins in a setting of relational employment contracts with public and private information about a permanent productivity shock. If the firm cannot formally commit to conditional payments, the total amount of bonuses that the firm can pay to its employees is constrained by the expected discounted profits the firm can earn if the relationship continues in the next periods. Naturally, when the firm and workers are sufficiently patient, these constraints are not binding and the first-best action profile is attainable. For intermediate values of the discount factor, however, the lack of commitment implies that the optimal self-enforcing relational plan is inefficient and that at least one of the margins is distorted relative to the first-best.

When information about the firm's productivity is public, we have shown that conditioning the number of employees and their effort only to the productivity is sufficient for optimality, as all deviations are observed. Moreover, when the probability that productivity will fall is sufficiently high and for intermediate values of δ, we observe that the level of effort exerted by employees in a high-productivity state is distorted downwards relative to the first-best level. In this case, depending on the curvature of the production and cost functions, the firm may optimally hire more employees than it would if contracts were formal. Intuitively, since increasing the number of employees does not tighten workers' individual incentive compatibility constraints, the firm may use hiring as an adjustment instrument. As productivity falls, the firm sets the number of employees and their effort to the low-productivity first-best level, a dynamic that is characterized by a layoff. Interestingly, though, since the first-best effort level is the same for any productivity, this implies that we would observe the remaining employees exerting more effort following the layoff. This increase is driven by the inefficiently low level of effort with high productivity, not an inefficiently high level of effort after productivity falls.

If the firm's productivity is private, a firm in a high-productivity state can falsely claim a bleak future and pay its employees smaller bonuses. Thus, private information generates an informational rent for a high-productivity firm, which, in turn, tightens its incentive compatibility constraints. For that reason, in this case, the optimal self-enforcing relational plan is history-dependent and the optimal action profile depends on the distance to the last high-productivity state announced. For intermediate values of δ, in a high-productivity state, the firm distorts the effort exerted downwards relative to first-best and may distort the number of employees downwards or upwards, depending on the cost and production functions, similarly to the complete information case. Moreover, when productivity falls, the firm sets the effort level back to the first-best, thus increasing the effort of the remaining employees. In contrast, however, optimality requires now a bigger layoff, with the number of employees falling below the low-productivity first-best level. Additionally, over time, the firm hires new employees and the number of employees approaches first-best in the long run. This result suggests that the intensive margin is distorted only due to commitment issues, while the extensive margin is distorted also due to informational asymmetries.

For further research, we aim to explore optimal contractual structures in this environment. An important assumption in our model is that the contracts are multilateral, that is when the contracts of all employees are dissolved when a deviation from the firm is observed. However, firms often contract bilaterally with at least some of their employees. As observed in Levin (2002), multilateral contracts allow the firm to commit to larger bonuses, as punishments to deviations are also larger. On the other hand, they offer less flexibility. If there is some communication problem or uncertainty about the firm's reputation when laying off employees, it may be in the firm's best interest to structure the contracts of its employees to be sufficiently flexible while also generating commitment. This can be done, for instance, by having a share of "fixed" employees with multilateral contracts and a share of employees with bilateral contracts that may be laid off in the case of a recession. We believe the simple framework in this paper allows us to analyze how this optimal contractual structure depends on the primitives of the economy. such as p, δ, and the production and cost functions.

Another interesting extension is to allow for more general relationships between effort and the number of employees in the production function. Including worker heterogeneity in the model might also lead us to interesting results. Nevertheless, incorporating worker heterogeneity while maintaining some anonymity is a technical and conceptual challenge.

A Appendix - Complete information

A.1 Proof of Lemma 1

Proof. Let $\tilde{\mathcal{P}} = \left(\tilde{N}_t(\theta^t), \tilde{e}_t(i,\theta^t), \tilde{w}_t(i,\theta^t), \tilde{b}^h_t(i,\theta^t), \tilde{b}^l_t(i,\theta^t)\right)$ be an optimal self-enforcing relational plan.

1. Let $\tilde{r}_t(i,\theta^t) = \tilde{U}_t(i,\theta^t) - \overline{u}$ be worker i's rent at θ^t. Suppose that there exist $t_1 \geq 0$ and $\theta^{t_1} = (\theta^{t_1-1}, \theta^k)$ such that $\int_0^{\tilde{N}_{t_1}(\theta^{t_1})} \tilde{r}_{t_1}(i,\theta^{t_1})di > 0$. By Assumption 2, an agent who is employed at t_1, following history θ^{t_1}, either must have been employed at $t_1 - 1$, following θ^{t_1-1}, or must have been hired at t_1. Let $\tilde{I}_{t_1-1} = \left\{i : i \leq \min\left\{\tilde{N}_{t_1-1}(\theta^{t_1-1}), \tilde{N}_{t_1}(\theta^{t_1})\right\}\right\}$, the set of agents who were employed at $t_1 - 1$, and $\tilde{I}_{t_1} = \left[0, \tilde{N}_{t_1}(\theta^{t_1})\right] \setminus \tilde{I}_{t_1-1}$, the set of agents hired at t_1.

   Consider a new plan $\mathcal{P}$ with the same action profile of $\tilde{\mathcal{P}}$ and contracts $\left(w_t(i,\theta^t), b^h_t(i,\theta^t), b^l_t(i,\theta^t)\right)$ equal to those in $\tilde{\mathcal{P}}$ but with the following changes:
   (i) for every $i \leq \tilde{N}_{t_1}(\theta^{t_1})$, $w_{t_1}(i,\theta^{t_1}) = \tilde{w}_{t_1}(i,\theta^{t_1}) - \left(\tilde{U}_{t_1}(i,\theta^{t_1}) - \overline{u}\right)$; and
   (ii) $$b^k_{t_1-1}(i,\theta^{t_1-1}) = \begin{cases} \tilde{b}^k_{t_1-1}(i,\theta^{t_1-1}) + \dfrac{\delta}{(1-\delta)}\left(1 - \tilde{\xi}_{t_t-1}(\theta^{t_1})\right)\left(\tilde{U}_{t_1}(i,\theta^{t_1}) - \overline{u}\right) & i \in \tilde{I}_{t_1-1} \\ \tilde{b}^k_{t_1-1}(i,\theta^{t_1-1}) & i \in \tilde{I}_{t_1} \end{cases}$$
   For those agents in $\tilde{I}_{t_1}$, only the wage at θ^{t_1} is changed, and the continuation payoff is reduced to $\overline{u}$, so no constraint is broken. For those in $\tilde{I}_{t_1-1}$, note that $(1-\delta)b^k_{t_1-1}(i,\theta^{t_1-1}) + \delta\left(1 - \tilde{\xi}_{t_t-1}(\theta^{t_1})\right)U_{t_1}(i,\theta^{t_1}) = (1-\delta)\tilde{b}^k_{t_1-1}(i,\theta^{t_1-1}) + \left(1 - \tilde{\xi}_{t_t-1}(\theta^{t_1})\right)\delta\tilde{U}_{t_1}(i,\theta^{t_1})$ and that the continuation payoffs and bonuses remain the same for histories of length $\tau < t_1$, hence the (IC^w) and (IR^w) constraints still hold for those periods. At t_1, the continuation payoff is reduced to $\overline{u}$, so the (IR^w) is also satisfied.

   Note that the new continuation profits satisfy $\Pi_\tau(\theta^\tau) \geq \tilde{\Pi}_\tau(\theta^\tau)$ for all histories of length $\tau < t_1$ (in particular $\Pi_0(\theta^h) = \tilde{\Pi}_0(\theta^h)$) and $\Pi_\tau(\theta^\tau) = \tilde{\Pi}_\tau(\theta^\tau)$ for those with length $\tau > t_1$, and $\Pi_{t_1}(\theta^{t_1}) = \tilde{\Pi}_{t_1}(\theta^{t_1}) + \int_0^{N_{t_1}(\theta^{t_1})} r_{t_1}(i,\theta^{t_1})di > \tilde{\Pi}_{t_1}(\theta^{t_1})$; moreover, $-(1-\delta)\int_0^{\tilde{N}_\tau(\theta^\tau)} b^s_\tau(i,\theta^\tau)di + \delta\Pi_{\tau+1}(\theta^\tau,\theta^s) \geq -(1-\delta)\int_0^{\tilde{N}_\tau(\theta^\tau)} \tilde{b}^s_\tau(i,\theta^\tau)di + \delta\tilde{\Pi}_{\tau+1}(\theta^\tau,\theta^s)$, $s = h, l$, for all histories. Therefore, the new plan also satisfies the constraints (IR^f), $(IC^{f,h})$, and $(IC^{f,l})$. If I_{t_1}has positive mass and $\int_{I_{t_1}} r_{t_1}(i,\theta^{t_1})di > 0$, then the new plan strictly increases the firm's profits and we have a contradiction with the optimality of $\tilde{P}$. If that is not the case, then the inequality above actually binds for all histories. Hence, we can proceed to make the same changes for all histories of length $t_1 - 1$ and smaller.

2. Given 1., the (IC^w) constraint for a worker i can be written, for all $t \geq 0$ and θ^t, as

$$\left(1 - P(\theta^l|\theta_t)\right) \tilde{b}_t^h(i,\theta^t) + P(\theta^l|\theta_t)\tilde{b}_t^l(i,\theta^t) \geq c(\tilde{e}_t(i,\theta^t))$$

Suppose there are t_1 and θ^{t_1} such that

$$\left(1 - P(\theta^l|\theta_{t_1})\right) \int_0^{\tilde{N}_{t_1}(\theta^{t_1})} \tilde{b}_{t_1}^h(i,\theta^{t_1})di + P(\theta^l|\theta_{t_1}) \int_0^{\tilde{N}_{t_1}(\theta^{t_1})} \tilde{b}_{t_1}^l(i,\theta^t)di$$
$$> \int_0^{\tilde{N}_{t_1}(\theta^{t_1})} c(\tilde{e}_{t_t}(i,\theta^{t_1}))di$$

This implies that there is a mass of workers $\tilde{I}$ for whom the (IC^w) holds with strictly inequality, and that either $\left(1 - P(\theta^l|\theta_{t_1})\right) \int_{\tilde{I}} \tilde{b}_{t_1}^h(i,\theta^{t_1})di > 0$ or $P(\theta^l|\theta_{t_1}) \int_{\tilde{I}} \tilde{b}_{t_1}^l(i,\theta^t)di > 0$. Then there exists a function $\tilde{\epsilon}(i) > 0$ such that, for each $i \in I$, we can reduce either $\tilde{b}_{t_1}^h(i,\theta^{t_1})$ by $\dfrac{\tilde{\epsilon}(i)}{\left(1 - P(\theta^l|\theta_{t_1})\right)}$ or $\tilde{b}_{t_1}^l(i,\theta^{t_1})$ by $\dfrac{\tilde{\epsilon}(i)}{P(\theta^l|\theta_{t_1})}$, while keeping the (IC^w) satisfied for all workers, and increase $\tilde{w}_{t_1}(i,\theta^{t_1})$ by $\tilde{\epsilon}(i)$. This change keeps every worker's utility unchanged and the firm's expected profits the same but loosens at least one of the (IC^f) constraints.

3. This condition follows directly from the convexity of the cost function c. By 1., we can write the (IC^w) constraints for workers $i \leq \tilde{N}_t(\theta^t)$ collectively as

$$P(\theta^h|\theta_t) \int_0^{\tilde{N}_t(\theta^t)} \tilde{b}_t^h(i,\theta^t)di + (1 - P(\theta^h|\theta_t)) \int_0^{\tilde{N}_T(\theta^t)} \tilde{b}_t^l(i,\theta^t)di \geq \int_0^{\tilde{N}_T(\theta^t)} c(\tilde{e}_t(i,\theta^t))di$$

Suppose there are t_1 and θ^{t_1} such that $\tilde{e}_{t_1}(\cdot,\theta^{t_1})$ is not constant and let

$$e_{t_1}(i,\theta^{t_1}) = e_{t_1}(\theta^{t_1}) = \frac{1}{\tilde{N}_{t_1}(\theta^{t_1})} \int_0^{\tilde{N}_{t_1}(\theta^{t_1})} \tilde{e}_{t_1}(i,\theta^{t_1})di$$

Clearly, $\theta_{t_1}\psi\left(\tilde{N}_{t_1}(\theta^{t_1})e_{t_1}(\theta^{t_1})\right) = \theta_{t_1}\psi\left(\int_0^{\tilde{N}_{t_1}(\theta^{t_1})} \tilde{e}_{t_1}(i,\theta^{t_1})di\right)$. Additionally, by convexity

$$\tilde{N}_{t_1}(\theta^{t_1})c(e_{t_1}(\theta^{t_1})) < \int_0^{\tilde{N}_{t_1}(\theta^{t_1})} c(\tilde{e}_{t_1}(i,\theta^{t_1}))di$$

For workers $i \leq \tilde{N}_{t_1}(\theta^{t_1})$, let $b_{t_1}^s(i,\theta^{t_1}) = \tilde{b}_{t_1}^s(i,\theta^{t_1}) - (c(\tilde{e}_{t_1}(i,\theta^{t_1})) - c(e_{t_1}(\theta^{t_1})))$ for both

$s = h, l$, so the each individual (IC^w) constraint still holds. Moreover, note that

$$\int_0^{\tilde{N}_{t_1}(\theta^{t_1})} b^s_{t_1}(i, \theta^{t_1})di < \int_0^{\tilde{N}_{t_1}(\theta^{t_1})} \tilde{b}^s_t(i, \theta^{t_1})di$$

for $s = h, l$. Therefore, these changes not only increase the continuation profits but also loosen the $(IC^{f,h})$ and $(IC^{f,l})$ constraints for histories of length t_1 and shorter.

□

A.2 Proof of Proposition 1

Proof. Let $\tilde{\mathcal{P}} = \left(\tilde{N}_t(\theta^t), \tilde{e}_t(\theta^t), \tilde{w}_t(\theta^t), \tilde{b}^h_t(\theta^t), \tilde{b}^l_t(\theta^t)\right)_{t\geq 0}$ be an optimal self-enforcing relational plan that satisfies the properties in Lemma 1 and $\tilde{\Pi}_0(\theta^h)$ the ex-ante expected profits in $\tilde{\mathcal{P}}$. By Lemma 1, the (IR^w) and (IC^w) constraints hold with equality. Note that, given complete information, in an optimal self-enforcing relational plan it must hold that $\tilde{\Pi}_0(\theta^h) = \tilde{\Pi}_{t+1}(\theta^t, \theta^h)$ for all $t \geq 0$, $(\theta^t, \theta^h) \in \Theta^{t+1}$. It is clear that $\tilde{\Pi}_0(\theta^h) \geq \tilde{\Pi}_{t+1}(\theta^t, \theta^h)$, since if $\tilde{\Pi}_0(\theta^h) < \tilde{\Pi}_{t+1}(\theta^t, \theta^h)$, we could just substitute $\tilde{\mathcal{P}}$ with the continuation plan after (θ^t, θ^h), strictly increasing the firm's ex-ante profits. On the other hand, suppose $\tilde{\Pi}_0(\theta^h) > \tilde{\Pi}_{t+1}(\theta^t, \theta^h)$. Consider substituting the continuation action profile $\left(\tilde{N}_\tau(\theta^\tau), \tilde{e}_\tau(\theta^\tau)\right)_{\tau\geq t+1}$ with the one in $\tilde{\mathcal{P}}$, $\left(\tilde{N}_t(\theta^t), \tilde{e}_t(\theta^t)\right)_{t\geq 0}$, increasing $\tilde{\Pi}_{t+1}(\theta^t, \theta^h)$ to $\tilde{\Pi}_0(\theta^h)$. In this new plan, (IR^w) and (IC^w) hold with equality by construction. Moreover, this change increases continuation profits for some histories of length shorter than t, thereby loosening the $(IC^{f,h})$ and (IR^f) constraints, while also strictly increasing $\tilde{\Pi}_0$. Hence, we have a contradiction. As the same argument can be made for histories ending with θ^l, we also have that $\tilde{\Pi}_{t+1}(\theta^t, \theta^l) = \tilde{\Pi}_{t'+1}(\theta^{t'}, \theta^l)$ for all $(\theta^t, \theta^l) \in \Theta^{t+1}$ and $(\theta^{t'}, \theta^l) \in \Theta^{t'+1}$. Therefore, by replicating the contracts and action profiles at (θ^h) and (θ^h, θ^l), we can construct an optimal stationary Markov self-enforcing relational plan $\mathcal{P}$ that generates the same ex-ante profits as in $\tilde{\mathcal{P}}$. □

A.3 Proof of Proposition 2

Proof. Let

$$\phi^l_{ci}(\delta) = \delta\pi^{fb,l} - (1-\delta)N^{fb,l}c(e^{fb,l})$$

and

$$\phi^h_{ci}(\delta) = \delta\pi^{fb,h} - (1-\delta)N^{fb,h}c(e^{fb,h})$$

Let

$$\delta^l_{ci} = \frac{N^{fb,l}c(e^{fb,l})}{\theta^l\psi(N^{fb,l}e^{fb,l}) - N^{fb,l}\overline{u}} \in (0,1)$$

and

$$\delta^h_{ci} = \frac{N^{fb,h}c(e^{fb,h})}{\theta^h\psi(N^{fb,h}e^{fb,h}) - N^{fb,h}\overline{u}} \in (0,1)$$

Note that both functions $\phi^h_{ci}(\delta)$ and $\phi^l_{ci}(\delta)$ are continuous and strictly increasing functions of δ, and that $\phi^l_{ci}(\delta^l_{ci}) = 0$ and $\phi^h_{ci}(\delta^h_{ci}) = 0$. Thus, δ^l_{ci} is the lowest discount factor such that $(DE^{f,l})$ holds with $(N^{fb,l}, e^{fb,l})$, and δ^h_{ci} is analogous to δ^l_{ci} but for $\theta = \theta^h$ and $(N^{fb,h}, e^{fb,h})$. Note that, since $e^{fb,h} = e^{fb,l}$, we have

$$\begin{aligned}\frac{\delta^h_{ci}}{\delta^l_{ci}} &= \frac{\dfrac{N^{fb,h}c(e^{fb,h})}{\theta^h\psi(N^{fb,h}e^{fb,h}) - N^{fb,h}\overline{u}}}{\dfrac{N^{fb,l}c(e^{fb,l})}{\theta^l\psi(N^{fb,l}e^{fb,l}) - N^{fb,l}\overline{u}}} \\ &= \frac{\dfrac{\theta^l\psi(N^{fb,l}e^{fb,l})}{N^{fb,l}} - \overline{u}}{\dfrac{\theta^h\psi(N^{fb,h}e^{fb,h})}{N^{fb,h}} - \overline{u}}\end{aligned}$$

so

$$\begin{aligned}\delta^h_{ci} < \delta^l_{ci} &\Leftrightarrow \frac{\theta^l\psi(N^{fb,l}e^{fb,l})}{N^{fb,l}} - \overline{u} < \frac{\theta^h\psi(N^{fb,h}e^{fb,h})}{N^{fb,h}} - \overline{u} \\ &\Leftrightarrow \frac{\theta^l\psi(N^{fb,l}e^{fb,l})}{N^{fb,l}} < \frac{\theta^h\psi(N^{fb,h}e^{fb,h})}{N^{fb,h}}\end{aligned}$$

Fix $p \in (0,1)$. Let

$$\phi^h_{ci,de}(\delta) = \frac{(1-p)(1-\delta)}{1-(1-p)\delta}\pi^{fb,h} + \frac{p}{1-(1-p)\delta}\pi^{fb,l} - \frac{1-\delta}{\delta}N^{fb,h}c(e^{fb,h})$$

We have that $\lim_{\delta\to 0}\phi^h_{ci,de}(\delta) = -\infty$ and $\lim_{\delta\to 1}\phi^h_{ci,de}(\delta) = \pi^{fb,l} > 0$. Thus, there exist $\delta^{1,h}_{ci}(p) \in (0,1)$ such that $\phi^h_{ci,de}(\delta^{1,h}_{ci}(p)) = 0$. Moreover, since $\pi^{fb,h} > \pi^{fb,l}$, $\lim_{\delta\uparrow\frac{1}{1-p}}\phi^h_{ci,de}(\delta) = -\infty$; together with the fact that $\phi^h_{ci,de}$ has up to two roots, this implies that the larger one has to lie above 1, so only the smaller one, $\delta^{1,h}_{ci}(p)$, lies between 0 and 1. Thus, for $\delta \in (0,1)$, $\phi^h_{ci,de}(\delta) > 0$ if, and only if, $\delta > \delta^{1,h}_{ci}(p)$. Since $\dfrac{(1-p)(1-\delta)}{1-(1-p)\delta}\pi^{fb,h} + \dfrac{p}{1-(1-p)\delta}\pi^{fb,l}$ is an average between $\pi^{fb,h}$ and $\pi^{fb,l} < \pi^{fb,h}$, we have that $\phi^h_{ci}(\delta) > \phi^h_{ci,de}(\delta)$ for all $\delta \in (0,1)$, which implies that $\delta^{1,h}_{ci}(p) > \delta^h_{ci}$. Note that $\lim_{p\to 0}\phi^h_{ci,de}(\delta) = \phi^h_{ci}(\delta)$ for all $\delta \in (0,1)$, so

$\lim_{p\to 0}\delta_{ci}^{1,h}(p) = \delta_{ci}^{h}$; moreover, $\lim_{p\to 1}\phi_{ci,de}^{h}(\delta) = \pi^{fb,l} - \frac{1-\delta}{\delta}N^{fb,h}c(e^{fb,h}) < \phi_{ci}^{l}(\delta)$ for all $\delta \in (0,1)$, so $\lim_{p\to 1}\delta_{ci}^{1,h}(p) > \delta_{ci}^{l}$. By the Implicit Function Theorem, $\delta_{ci}^{1,h}(p)$ is a continuous functions of p, and since $\phi_{ci,de}^{h}(\delta)$ is strictly decreasing in p, $\delta_{ci}^{1,h}(p)$ is strictly increasing. If $\delta_{ci}^{h} \geq \delta_{ci}^{l}$, than for all $p \in (0,1]$ we have $\delta_{ci}^{1,h}(p) > \delta_{ci}^{l}$, while $\delta_{ci}^{1,h}(0) = \delta_{ci}^{h}$. If $\delta_{ci}^{h} < \delta_{ci}^{l}$, then there exists $p^\star \in (0,1)$ such that $\delta_{ci}^{1,h}(p) > \delta_{ci}^{l}$ if, and only if, $p > p^\star$.

Finally, whenever $\delta_{ci}^{h} < \delta_{ci}^{l}$ and $p < p^\star$ (so $\delta_{ci}^{1,h}(p) < \delta_{ci}^{l}$), let $(N^l(\delta), e^l(\delta))$ be the low-state action profile in the solution to the maximization problem for $\delta \in (\delta_{ci}^{1,h}(p), \delta_{ci}^{l})$ and $\pi^l(\delta)$ the low-state profits at this solution. As $\delta < \delta_{ci}^{l}$, we know that $(DE^{f,l})$ must bind and that $\pi^l(\delta) < \pi^{fb,l}$. Let $\phi_{ci,de}^{l}(\delta) = \frac{(1-p)(1-\delta)}{1-(1-p)\delta}\pi^{fb,h} + \frac{p}{1-(1-p)\delta}\pi^l(\delta) - \frac{1-\delta}{\delta}N^{fb,h}c(e^{fb,h})$, so $\phi_{ci,de}^{l}(\delta) < \phi_{ci,de}^{h}(\delta)$. Note that, since $\pi^l(\delta) \uparrow \pi^{fb,l}$ as $\delta \uparrow \delta^l$, $|\phi_{ci,de}^{h}(\delta) - \phi_{ci,de}^{l}(\delta)| \downarrow 0$ as $\delta \uparrow \delta^l$. Since $\delta_{ci}^{1,h}(p) < \delta_{ci}^{l}$, this implies that there exists $\delta_{ci}^{2,h}(p) \in (\delta_{ci}^{1,h}(p), \delta_{ci}^{l})$ such that $\phi_{ci,de}^{l}(\delta) > 0$ if, and only if, $\delta \in (\delta_{ci}^{2,h}(p), \delta_{ci}^{l})$. Note that $\delta_{ci}^{2,h}(p)$ is only defined for $p \in (0, p^\star)$. For completeness, let $\delta_{ci}^{2,h}(p) = \delta_{ci}^{1,h}(p)$ for $p \in (p^\star, 1)$.

The Lagrangian of the problem is

$$\begin{aligned}
\mathcal{L} &= \frac{(1-\delta)}{1-(1-p)\delta}\pi^h + \frac{p\delta}{1-(1-p)\delta}\pi^l \\
&+ \lambda^h\left[-(1-\delta)N^h c(e^h) + \frac{(1-p)(1-\delta)\delta}{1-(1-p)\delta}\pi^h + \frac{\delta p}{1-(1-p)\delta}\pi^l\right] \\
&+ \lambda^l\left[-(1-\delta)N^l c(e^l) + \delta\pi^l\right]
\end{aligned}$$

The first-order conditions are

$$\frac{\partial\mathcal{L}}{\partial e^h} = (1-\delta)\left[\frac{1+\lambda^h(1-p)\delta}{1-(1-p)\delta}\right]\left[\theta^h\psi'\left(N^h e^h\right)N^h - c'(e^h)N^h\right] - \lambda^h(1-\delta)c'(e^h)N^h = 0 \quad (7)$$

$$\frac{\partial\mathcal{L}}{\partial e^l} = \left[\frac{p\delta}{1-(1-p)\delta}(1+\lambda^h) + \delta\lambda^l\right]\left[\theta^l\psi'\left(N^l e^l\right)N^l - c'(e^l)N^l\right] - \lambda^l(1-\delta)c'(e^l)N^l = 0 \quad (8)$$

and

$$\frac{\partial\mathcal{L}}{\partial N^h} = (1-\delta)\left[\frac{1+\lambda^h(1-p)\delta}{1-(1-p)\delta}\right]\left[\theta^h\psi'\left(N^h e^h\right)e^h - \overline{u} - c(e^h)\right] - \lambda^h(1-\delta)c(e^h) = 0 \quad (9)$$

$$\frac{\partial\mathcal{L}}{\partial N^l} = \left[\frac{p\delta}{1-(1-p)\delta}(1+\lambda^h) + \delta\lambda^l\right]\left[\theta^l\psi'\left(N^l e^l\right)e^l - \overline{u} - c(e^l)\right] - \lambda^l(1-\delta)c(e^l) = 0 \quad (10)$$

For $\delta \geq \max\left\{\delta^{l}_{ci}, \delta^{1,h}_{ci}(p)\right\}$, we clearly have $\lambda^l = \lambda^h = 0$, $(N^l, e^l) = (N^{fb,l}, e^{fb,l})$, and $(N^h, e^h) = (N^{fb,h}, e^{fb,h})$.

For $p > p^\star$ and $\delta \in (\delta^{l}_{ci}, \delta^{1,h}_{ci}(p))$, we have $\phi_l(\delta) > 0$ and $\phi_3(\delta) < 0$. This, then, implies that $(N^l, e^l) = (N^{fb,l}, e^{fb,l})$, since setting actions this way increases ex-ante profits and also loosens $(DE^{f,h})$, while $(DE^{f,h})$ binds. In this case, $\lambda^h > 0$ and $\lambda^l = 0$, thus from 7 and 9 we get

$$\theta^h \psi'\left(N^h e^h\right) = \left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right] c'(e^h) \tag{11}$$

and

$$\theta^h \psi'\left(N^h e^h\right) e^h = \overline{u} + \left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right] c(e^h) \tag{12}$$

11 and 12 imply

$$\overline{u} = \left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right]\left[c'(e^h)e^h - c(e^h)\right] > c'(e^h)e^h - c(e^h)$$

since $1 + \dfrac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h > 1$. Now, $e^h < e^{fb,h}$ comes from the fact that $c'(e)e - c(e)$ is strictly increasing and $c'(e^{fb,h})e^{fb,h} - c(e^{fb,h}) = \overline{u}$. On the other hand, by substituting $\left[1 + \dfrac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right] = \dfrac{\theta^h \psi'\left(N^h e^h\right)}{c'(e^h)}$ into 12, we get

$$\theta^h \psi'\left(N^h e^h\right)\left[e^h - \frac{c(e^h)}{c'(e^h)}\right] = \overline{u}$$

The derivative of the LHS with respect to e determines whether $N^h \lesseqgtr N^{fb,h}$: if it is positive, then $N^h < N^{fb,h}$; if it is negative, then $N^h > N^{fb,h}$. Additionally, 8 and 10 imply $(N^l, e^l) = (N^{fb,l}, e^{fb,l})$.

For $p < p^\star$ and $\delta \in (\delta^{1,2}_{ci}(p), \delta^{l}_{ci})$, we have that $(DE^{f,l})$ binds, but $(DE^{f,h})$ does not. In this case, $\lambda^h = 0$ and $\lambda^l > 0$, thus 7 and 9 imply $(N^h, e^h) = (N^{fb,h}, e^{fb,h})$. Moreover, 8 and 10 imply

$$\theta^l \psi'\left(N^l e^l\right) = \left[1 + \frac{1-(1-p)\delta}{p\delta + \lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l\right] c'(e^l) \tag{13}$$

and

$$\theta^l \psi'\left(N^l e^l\right) e^l = \overline{u} + \left[1 + \frac{1-(1-p)\delta}{p\delta + \lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l\right] c(e^l) \tag{14}$$

13 and 14 imply

$$\overline{u}=\left[1+\frac{1-(1-p)\delta}{p\delta+\lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l\right]\left[c'(e^l)e^l-c(e^l)\right]>c'(e^l)e^l-c(e^l)$$

since $1+\frac{1-(1-p)\delta}{p\delta+\lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l>1$. This, in turn, implies $e^l<e^{fb,l}$. Additionally, by substituting

$$\left[1+\frac{1-(1-p)\delta}{p\delta+\lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l\right]=\frac{\theta^l\psi'\left(N^le^l\right)}{c'(e^l)}$$

into 14, we get

$$\theta^h\psi'\left(N^le^l\right)\left[e^l-\frac{c(e^l)}{c'(e^l)}\right]=\overline{u}$$

The derivative of the LHS with respect to e determines whether $N^l\lesseqgtr N^{fb,l}$: if it is positive, then $N^l<N^{fb,l}$; if it is negative, then $N^l>N^{fb,l}$.

For $p<p^\star$ and $\delta<\min\left\{\delta_{ci}^{2,h}(p),\delta_{ci}^l\right\}=\delta_{ci}^{2,h}(p)$ or $p>p^\star$ and $\delta<\min\left\{\delta_{ci}^l,\delta_{ci}^{1,h}(p)\right\}=\delta_{ci}^l$, both $(DE^{f,l})$ and $(DE^{f,h})$ bind, so $\lambda^h,\lambda^l>0$. In these cases, 11 and 12 imply that $e^h<e^{fb,h}$ and $N^h\lesseqgtr N^{fb,h}$, and

$$\theta^l\psi'\left(N^le^l\right)=\left[1+\frac{1-(1-p)\delta}{(1+\lambda^h)p\delta+\lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l\right]c'(e^l) \tag{15}$$

and

$$\theta^l\psi'\left(N^le^l\right)e^l=\overline{u}+\left[1+\frac{1-(1-p)\delta}{(1+\lambda^h)p\delta+\lambda^l\delta(1-(1-p)\delta)}(1-\delta)\lambda^l\right]c(e^l) \tag{16}$$

imply $e^l<e^{fb,l}$ and $N^l\lesseqgtr N^{fb,l}$. On both cases, $N^k\lesseqgtr N^{fb,k}$ depends on whether $\theta^k\psi'(Ne)\left[e-\frac{c(e)}{c'(e)}\right]$ is an increasing or decreasing function of e for a given N, just like in the cases above. □

B Appendix - Private information

B.1 Proof of Lemma 3

Proof. Let $\tilde{\mathcal{P}}=\left(\tilde{N}_t(\theta^t),\tilde{e}_t(\theta^t),\tilde{w}_t(\theta^t),\tilde{b}_t^h(\theta^t),\tilde{b}_t^l(\theta^t)\right)_{t\geq0}$ be an optimal self-enforcing relational plan.

1. Suppose to the contrary that this is not the case and consider the changes proposed in the proof of Lemma 1. The only problem that might arise when implementing those changes

happens when $\int_{I_{t_1}} r_{t_1}(i,\theta^{t_1})di > 0$, that is to say, when there is a positive mass of workers that became employed at t_1 with a positive rent. In this case, the reduction in wages $\tilde{w}_{t_1}(i,\theta^{t_1})$, $i \in I_{t_1}$, simultaneously loosens the (TT^k) constraint and tightens the (TT^{-k}) constraint since it strictly increases continuation profits from t_1 onward without increasing the bonuses payed at θ^{t_1-1}. Thus, the new plan might not be self-enforcing. To circumvent this possible problem, we can redistribute the excess profits $\int_{I_{t_1}} r_{t_1}(i,\theta^{t_1})di$ generated by the reduction of wages at θ^{t_1} to the bonuses of workers employed at θ^{t_1-1}, provided that $-(1-\delta)\int_0^{\tilde{N}_{t_1-1}(\theta^{t_1-1})} b^k_{t_1-1}(\theta^{t_1-1})di + \delta\Pi_{t_1}(\theta^{t_1}) = -(1-\delta)\int_0^{\tilde{N}_{t_1-1}(\theta^{t_1-1})} \tilde{b}^k_{t_1-1}(\theta^{t_1-1})di + \delta\tilde{\Pi}_{t_1}(\theta^{t_1})$. This new change keeps unchanged all of the firm's constraints, while loosening the agents'. We then proceed to make the same alterations for all histories of length $t_1 - 1$ or shorter.

2. The proof follows Lemma 1.3 without any changes. It is only important to notice that the both conditional payments are altered by the same amount, so the changes does affect neither the (TT^h) constraint nor the (TT^l) constraint.

□

B.2 Proof of Proposition 3

Proof. Let $\tilde{\mathcal{P}} = \left(\tilde{N}_t(\theta^t), \tilde{e}_t(\theta^t), \tilde{w}_t(\theta^t), \tilde{b}^h_t(\theta^t), \tilde{b}^l_t(\theta^t)\right)_{t\geq 0}$ be an optimal self-enforcing relational plan that satisfies the properties in Lemma 3 and $\tilde{\Pi}_0(\theta^h)$ the ex-ante expected profits in $\tilde{\mathcal{P}}$. Firstly, we have $\tilde{\Pi}_t(\theta^{t-1},\theta^h) \leq \tilde{\Pi}_0(\theta^h)$ for all t and $\theta^{t-1} \in \Theta^{t-1}$, since if $\tilde{\Pi}_t(\theta^{t-1},\theta^h) > \tilde{\Pi}_0(\theta^h)$ for some $(\theta^{t-1},\theta^h) \in \Theta^t$, we could substitute the original plan with the continuation plan after (θ^{t-1},θ^h), which is feasible given the fact that, by Lemma 3, inter-temporal incentives must be given only to the firm,[13] and all constraints hold for periods $\tau \geq t$.

Next, write $\tilde{\Pi}_0(\theta^h) = (1-\delta)\tilde{\pi}_0(\theta^h) + (1-p)\delta\tilde{\Pi}_1(\theta^h,\theta^h) + p\delta\tilde{\Pi}_1(\theta^h,\theta^l)$. We state that $\tilde{\Pi}_1(\theta^h,\theta^h) \geq \tilde{\Pi}_0(\theta^h)$. Suppose not, that is $\tilde{\Pi}_1(\theta^h,\theta^h) < \tilde{\Pi}_0(\theta^h)$. In this case,

$$\begin{aligned}\tilde{\Pi}_0(\theta^h) &= (1-\delta)\tilde{\pi}_0(\theta^h) + (1-p)\delta\tilde{\Pi}_1(\theta^h,\theta^h) + p\delta\tilde{\Pi}_1(\theta^h,\theta^l)\\ &< (1-\delta)\tilde{\pi}_0(\theta^h) + (1-p)\delta\tilde{\Pi}_0(\theta^h) + p\delta\tilde{\Pi}_1(\theta^h,\theta^l)\end{aligned}$$

implies $\tilde{\Pi}_0(\theta^h) < \frac{(1-\delta)}{1-(1-p)\delta}\tilde{\pi}_0(\theta^h) + \frac{p\delta}{1-(1-p)\delta}\tilde{\Pi}_1(\theta^h,\theta^l)$. Consider, then, a new plan in

[13] Since, by Lemma 3, $\tilde{U}_t(i,\theta^t) = \overline{u}$ for all $\theta^t \in \Theta^t$, we do not need to worry about the workers' constraints when making this substitution.

which we set

$$\left(N_t(\theta^t), e_t(\theta^t), w_t(\theta^t), b_t^h(\theta^t), b_t^l(\theta^t)\right) = \left(\tilde{N}_0(\theta^h), \tilde{e}_0(\theta^h), \tilde{w}_0(\theta^h), \tilde{b}_0^h(\theta^h), \tilde{b}_0^l(\theta^h)\right)$$

for all $\theta^t \in \Theta^t$ such that $n(\theta^t) = 0$ and substitute the continuation play after all histories $\theta^t \in \Theta^t$ such that $n(\theta^t) = 1$ with the original continuation play after (θ^h, θ^l). This new plan generates ex-ante profits equal to $\Pi_0(\theta^h) = \frac{(1-\delta)}{1-(1-p)\delta}\tilde{\pi}_0(\theta^h) + \frac{p\delta}{1-(1-p)\delta}\tilde{\Pi}_1(\theta^h, \theta^l) > \tilde{\Pi}_0(\theta^h)$. Moreover,

$$\begin{aligned}
\Pi_{t+1}^{dl}(\theta^t, \theta^h) &= \tilde{\Pi}_1^{dl}(\theta^h, \theta^h) \\
&= \tilde{\Pi}_1(\theta^h, \theta^l) \\
&+ (1-\delta)(\theta^h - \theta^l)\sum_{\tau=0}^{\infty}((1-p)\delta)^\tau \psi\left(\tilde{N}_{\tau+1}(\theta^h, \theta^l, \ldots, \theta^l)\tilde{e}_{\tau+1}(\theta^h, \theta^l, \ldots, \theta^l)\right)
\end{aligned}$$

for all $(\theta^t, \theta^h) \in \Theta^{t+1}$ and $\Pi_t(\theta^t, \theta^l) = \tilde{\Pi}_1(\theta^h, \theta^l)$ for all $\theta^t \in \Theta^t$ such that $n(\theta^t) = 0$. Now, in this new plan, the (IR^w), $(IC^{f,l})$, and (IC^w) constraints all hold by construction. Moreover, the (TT^h) constraint holds with strict inequality. We then have found a contradiction, since this new plan strictly increases the firm's ex-ante profits while satisfying all constraints of interest. □

B.3 Proof of Lemma 4

Proof. Let $\tilde{\mathcal{P}} = \left(\tilde{N}^h, \tilde{e}^h, \tilde{w}^h, \tilde{b}^{h,h}, \tilde{b}^{h,l}, \tilde{N}^l(n), \tilde{e}^l(n), \tilde{w}^l(n), \tilde{b}^l(n)\right)_{n\in\mathbb{N}}$ be an optimal self-enforcing relational plan that satisfies the properties in Lemma 3 and Proposition 3. Suppose that $\tilde{b}^{h,h} < \tilde{b}^{h,l}$. Consider a new relational plan $\mathcal{P}$ in which $b^{h,h} = b^{h,l} = (1-p)\tilde{b}^{h,h} + p\tilde{b}^{h,l}$ and $w^h = \overline{u}$, while the rest is kept as in $\tilde{\mathcal{P}}$. In this new plan, the ex-ante and continuation profits remain the same, while $(IC_h^{f,l})$ constraint is loosened. If, under the new plan, the (TT^h) constraint holds, the proof is finished. If it does not, it must be that $\tilde{\Pi}^h < \tilde{\Pi}^{dl}(1)$. In this case, we can substitute the continuation play after (θ^h, θ^h) with the continuation play after (θ^h, θ^l) while setting $b^h(\theta^n) = b^l(\theta^n) = c(\tilde{e}^l(n))$ for any $\theta^n \in \Theta^n$. This new plan strictly increases the firms ex-ante profits and satisfies all constraints in the relaxed problem; in particular, it satisfies the (TT^h) constraint with equality. Thus, we have a contradiction. □

B.4 Proof of Lemma 5

Proof. Let $\tilde{\mathcal{P}} = \left(\tilde{N}^h, \tilde{e}^h, \tilde{w}^h, \tilde{b}^{h,h}, \tilde{b}^{h,l}, \tilde{N}^l(n), \tilde{e}^l(n), \tilde{w}^l(n), \tilde{b}^l(n)\right)_{n\in\mathbb{N}}$ be an optimal self-enforcing relational plan that satisfies the properties in Lemmas 3 and 4, and Proposition 3. Firstly, $\tilde{b}^l(n) = c(\tilde{e}^l(n))$. If not, we could decrease $\tilde{b}^l(n)$ by $\epsilon > 0$ while increasing $\tilde{w}^l(n)$ by ϵ. This keeps the ex-ante and continuation profits the same but loosens the $(IC^{f,l})$ constraint. Proceeding, now suppose that $(1-p)\tilde{b}^{h,h} + p\tilde{b}^{h,l} > c(\tilde{e}^h)$. If $\tilde{b}^{h,h} \geq \tilde{b}^{h,l} > 0$, then we can reduce both $\tilde{b}^{h,h}$ and $\tilde{b}^{h,l}$ by some $\epsilon > 0$ while increasing $\tilde{w}^h$ by ϵ. This keeps the ex-ante and continuation profits the same but loosens the $(IC^{f,l})$ constraint. If $\tilde{b}^{h,l} = 0$, then we can decrease $\tilde{b}^{h,h} > 0$ by some $\epsilon > 0$ and increase $\tilde{w}^h$ by $(1-p)\epsilon$. This keeps the ex-ante and continuation profits the same but loosens the (TT^h) constraint. □

B.5 Proof of Proposition 4

Proof. Let

$$\phi^h_{pi}(\delta) = \delta\left[\pi^{fb,h} - (\theta^h - \theta^l)\psi(N^{fb,l}e^{fb,l})\right] - (1-\delta)N^{fb,h}c(e^{fb,h})$$

and

$$\phi^l_{pi}(\delta) = \delta\pi^{fb,l} - (1-\delta)N^{fb,l}c(e^{fb,l})$$

We have that

$$\begin{aligned}&\pi^{fb,h} > \theta^h\psi\left(N^{fb,l}e^{fb,l}\right) - N^{fb,l}\overline{u} - N^{fb,l}c(e^{fb,l})\\ \Rightarrow&\pi^{fb,h} - (\theta^h - \theta^l)\psi\left(N^{fb,l}e^{fb,l}\right) > \pi^{fb,l} > 0\end{aligned}$$

Thus, both ϕ^h_{pi} and ϕ^l_{pi} are strictly increasing functions of δ.

Let

$$\delta^h_{pi} = \frac{N^{fb,h}c(e^{fb,h})}{\theta^h\psi(N^{fb,h}e^{fb,h}) - N^{fb,h}\overline{u} - (\theta^h - \theta^l)\psi(N^{fb,l}e^{fb,l})}$$

and

$$\delta^l_{pi} = \frac{N^{fb,l}c(e^{fb,l})}{\theta^l\psi(N^{fb,l}e^{fb,l}) - N^{fb,l}\overline{u}} = \delta^l_{ci}$$

be the unique roots of $\phi^h_{pi}(\delta)$ and $\phi^l_{pi}(\delta)$, respectively. As both ϕ^h_{pi} and ϕ^l_{pi} are strictly increasing in δ, δ^h_{pi} and δ^l_{pi} are the lowest discount factors for which ϕ^h_{pi} and ϕ^l_{pi} are positive, respectively.

Note that we have

$$\begin{aligned}\delta^h_{pi} > \delta^l_{pi} \Leftrightarrow & \frac{\theta^l\psi(N^{fb,l}e^{fb})}{N^{fb,l}} > \frac{\theta^h\psi(N^{fb,h}e^{fb})}{N^{fb,h}} - \frac{(\theta^h-\theta^l)\psi(N^{fb,l}e^{fb})}{N^{fb,h}}\\ \Leftrightarrow & \frac{\theta^l\psi(N^{fb,l}e^{fb})}{N^{fb,l}} > \frac{\theta^h\psi(N^{fb,h}e^{fb}) - \theta^h\psi(N^{fb,l}e^{fb})}{N^{fb,h}-N^{fb,l}}\\ \Leftrightarrow & \frac{\theta^l}{\theta^h} > \frac{\frac{\psi(N^{fb,h}e^{fb})-\psi(N^{fb,l}e^{fb})}{N^{fb,h}-N^{fb,l}}}{\frac{\psi(N^{fb,l}e^{fb})}{N^{fb,l}}}\end{aligned}$$

where $e^{fb} = e^{fb,h} = e^{fb,l}$. For instance, one case in which the inequality above holds is when $\theta^h \approx \theta^l$, since the LHS of the inequality converges to 1 as $\theta^h \downarrow \theta^l$, but the RHS converges to $\frac{\psi'\left(N^{fb,l}e^{fb}\right)}{\frac{\psi(N^{fb,l}e^{fb})}{N^{fb,l}e^{fb}}} < 1$.

Next, fix $p \in (0,1)$ and let

$$\begin{aligned}\phi^h_{pi,de}(\delta) = & \frac{(1-p)(1-\delta)\delta}{1-(1-p)\delta}\left[\pi^{fb,h} - (\theta^h-\theta^l)\psi\left(N^{fb,l}e^{fb,l}\right)\right]\\ & + \frac{p\delta}{1-(1-p)\delta}\pi^{fb,l} - (1-\delta)N^{fb,h}c(e^{fb,h})\end{aligned}$$

$\phi^h_{pi,de}(\delta)$ is the $(DE^{f,h})$ constraint with $(N^h, e^h) = (N^{fb,h}, e^{fb,h})$ and $(N^l(n), e^l(n)) = (N^{fb,l}, e^{fb,l})$ for all $n \in \mathbb{N}$. Note that

$$\frac{(1-p)(1-\delta)\delta}{1-(1-p)\delta}\left[\pi^{fb,h} - (\theta^h-\theta^l)\psi\left(N^{fb,l}e^{fb,l}\right)\right] + \frac{p\delta}{1-(1-p)\delta}\pi^{fb,l}$$

is an average and that $\pi^{fb,h} - (\theta^h-\theta^l)\psi\left(N^{fb,l}e^{fb,l}\right) > \pi^{fb,l}$, as we showed, so $\phi^h_{pi,de}(\delta) < \phi^h_{pi}(\delta)$. Thus, if we let $\delta^h_{pi}(p)$ be the root of $\phi^h_{pi,de}(\delta)$ in $(0,1)$, we have that $\delta^h_{pi}(p) > \delta^h_{pi}$. Now, note that $\phi^h_{pi,de}(\delta)$ is strictly decreasing in p, with $\lim_{p\to 0}\phi^h_{pi,de}(\delta) = \phi^h_{pi}(\delta)$ and $\lim_{p\to 1}\phi^h_{pi,de}(\delta) = \delta\pi^{fb,l} - (1-\delta)N^{fb,h}c(e^{fb,h}) < \phi^l_{pi}(\delta)$. Therefore, if $\delta^h_{pi} \geq \delta^l_{pi}$, then for all $p \in (0,1]$, we have $\delta^h_{pi}(p) > \delta^l_{pi}$, while $\delta^h_{pi}(0) = \delta^h_{pi}$. If $\delta^h_{pi} < \delta^l_{pi}$, then there exists $p^\star \in (0,1)$ such that $\delta^h_{pi}(p) > \delta^l_{pi}$ if, and only if, $p > p^\star$.

The Lagrangian of the relaxed problem is

$$\begin{aligned}\mathcal{L} &= \frac{(1-\delta)}{1-(1-p)\delta}\pi^h + \frac{p(1-\delta)}{1-(1-p)\delta}\sum_{n=1}^{\infty}\delta^n\pi^l(n) \\ &+ \lambda^h\left[-(1-\delta)N^h c(e^h) + \frac{(1-p)(1-\delta)\delta}{1-(1-p)\delta}\pi^h + \frac{p(1-\delta)}{1-(1-p)\delta}\sum_{n=1}^{\infty}\delta^n\pi^l(n)\right. \\ &\qquad \left.-(1-\delta)(\theta^h-\theta^l)\sum_{n=1}^{\infty}((1-p)\delta)^n\psi\left(N^l(n)e^l(n)\right)\right] \\ &+ \sum_{n=1}^{\infty}\lambda^l(n)\left[-(1-\delta)N^l(n)c(e^l(n)) + (1-\delta)\sum_{i=n+1}^{\infty}\delta^{i-n}\pi^l(i)\right]\end{aligned}$$

The first-order-conditions are

$$\frac{\partial\mathcal{L}}{\partial e^h} = (1-\delta)\left[\frac{1+\lambda^h(1-p)\delta}{1-(1-p)\delta}\right]\left[\theta^h\psi'\left(N^h e^h\right)N^h - c'(e^h)N^h\right] - \lambda^h(1-\delta)c'(e^h)N^h = 0 \quad (17)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial e^l(n)} &= \frac{p(1-\delta)}{1-(1-p)\delta}(1+\lambda^h)\delta^n\left[\theta^l\psi'\left(N^l(n)e^l(n)\right)N^l(n) - c'(e^l(n))N^l(n)\right] \\ &-\lambda^h(1-\delta)(\theta^h-\theta^l)\,((1-p)\delta)^n\psi'\left(N^l(n)e^l(n)\right)N^l(n) \\ &-\lambda^l(n)(1-\delta)c'(e^l(n))N^l(n) \\ &+(1-\delta)\left[\sum_{i=1}^{n-1}\delta^{n-i}\lambda^l(i)\right]\left[\theta^l\psi'\left(N^l(n)e^l(n)\right)N^l(n) - c'(e^l(n))N^l(n)\right] = 0\end{aligned} \quad (18)$$

and

$$\frac{\partial\mathcal{L}}{\partial N^h} = (1-\delta)\left[\frac{1+\lambda^h(1-p)\delta}{1-(1-p)\delta}\right]\left[\theta^h\psi'\left(N^h e^h\right)e^h - \overline{u} - c(e^h)\right] - \lambda^h(1-\delta)c(e^h) = 0 \quad (19)$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial N^l(n)} &= \frac{p(1-\delta)}{1-(1-p)\delta}(1+\lambda^h)\delta^n\left[\theta^l\psi'\left(N^l(n)e^l(n)\right)e^l(n) - \overline{u} - c(e^l(n))\right] \\ &-\lambda^h(1-\delta)(\theta^h-\theta^l)\,((1-p)\delta)^n\psi'\left(N^l(n)e^l(n)\right)e^l(n) \\ &-\lambda^l(n)(1-\delta)c(e^l(n)) \\ &+(1-\delta)\left[\sum_{i=1}^{n-1}\delta^{n-i}\lambda^l(i)\right]\left[\theta^l\psi'\left(N^l(n)e^l(n)\right)e^l(n) - \overline{u} - c(e^l(n))\right] = 0\end{aligned} \quad (20)$$

with $\lambda^h, \lambda^l(n) \geq 0$.

Now, assume $\delta \in (\delta^l_{pi}, \delta^h_{pi}(p))$. We maximize the problem with $\lambda^l(n) = 0$ for all $n \in \mathbb{N}$, in which $\lambda^l(n)$ are the Lagrange multipliers of the $(IC^{f,l}_l)$ constraints, and show that the solution indeed satisfies the constraints. First, note that $\lambda^h = 0$ implies $\left(N^h, e^h\right) = \left(N^{fb,h}, e^{fb,h}\right)$, a contradiction with our assumption. Thus, $\lambda^h > 0$ and the $(DE^{f,h})$ constraint holds with equality.

From 17 and 19, we get

$$\theta^h \psi'\left(N^h e^h\right) = \left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right] c'(e^h) \tag{21}$$

and

$$\theta^h \psi'\left(N^h e^h\right) e^h = \overline{u} + \left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right] c(e^h) \tag{22}$$

21 and 22 imply

$$\overline{u} = \left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right]\left[c'(e^h)e^h - c(e^h)\right] > c'(e^h)e^h - c(e^h)$$

since $1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h > 1$. Now, $e^h < e^{fb,h}$ comes from the fact that $c'(e)e - c(e)$ is strictly increasing and $c'(e^{fb,h})e^{fb,h} - c(e^{fb,h}) = \overline{u}$. On the other hand, by substituting $\left[1 + \frac{1-(1-p)\delta}{1+\lambda^h(1-p)\delta}\lambda^h\right] = \frac{\theta^h\psi'\left(N^h e^h\right)}{c'(e^h)}$ into 22, we get

$$\theta^h \psi'\left(N^h e^h\right)\left[e^h - \frac{c(e^h)}{c'(e^h)}\right] = \overline{u}$$

The derivative of the LHS with respect to e determines whether $N^h \lesseqgtr N^{fb,h}$: if it is positive, then $N^h < N^{fb,h}$; if it is negative, then $N^h > N^{fb,h}$.

From 18 and 20, we get

$$\begin{aligned}&\left[\frac{p}{1-(1-p)\delta}(1+\lambda^h)\delta^n + \sum_{i=1}^{n-1}\delta^{n-i}\lambda^l(i)\right]\left[\theta^l\psi'\left(N^l(n)e^l(n)\right) - c'(e^l(n))\right] \\ &= \lambda^h(\theta^h - \theta^l)\left((1-p)\delta\right)^n \psi'\left(N^l(n)e^l(n)\right) + \lambda^l(n)c'(e^l(n))\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\left[\frac{p}{1-(1-p)\delta}(1+\lambda^h)\delta^n + \sum_{i=1}^{n-1}\delta^{n-i}\lambda^l(i)\right]\left[\theta^l\psi'\left(N^l(n)e^l(n)\right)e^l(n) - \overline{u} - c(e^l(n))\right] \\ &= \lambda^h(\theta^h - \theta^l)\left((1-p)\delta\right)^n \psi'\left(N^l(n)e^l(n)\right)e^l(n) + \lambda^l(n)c(e^l(n))\end{aligned} \tag{24}$$

Together, 23 and 24 imply

$$\overline{u} = \left[1 + \frac{\lambda^l(n)}{\frac{p}{1-(1-p)\delta}(1+\lambda^h)\delta^n + \sum_{i=1}^{n-1}\delta^{n-i}\lambda^l(i)}\right]\left[c'(e^l(n))e^l(n) - c(e^l(n))\right] \geq c'(e^l(n))e^l(n) - c(e^l(n)) \tag{25}$$

since $\left[1 + \frac{\lambda^l(n)}{\frac{p}{1-(1-p)\delta}(1+\lambda^h)\delta^n + \sum_{i=1}^{n-1}\delta^{n-i}\lambda^l(i)}\right] \geq 1$. Thus, $e^l(n) \leq e^{fb,l} = e^{fb,h}$. We also have

$$\theta^l\psi'\left(N^l(n)e^l(n)\right)\left[e^l(n) - \frac{c(e^l(n))}{c'(e^l(n))}\right] = \left[\frac{\frac{p}{1-(1-p)\delta}(1+\lambda^h) + \sum_{i=1}^{n-1}\delta^{-i}\lambda^l(i)}{\frac{p}{1-(1-p)\delta}(1+\lambda^h) + \sum_{i=1}^{n-1}\delta^{-i}\lambda^l(i) - \lambda^h(\theta^h-\theta^l)(1-p)^n}\right]\overline{u} > \overline{u} \tag{26}$$

Imposing $\lambda^l(n) = 0$ for all $n \in \mathbb{N}$, 25 implies $e^l(n) = e^{fb,l}$. Given that, we have $e^l(n) - \frac{c(e^l(n))}{c'(e^l(n))} = \frac{\overline{u}}{c'(e^{fb,l})}$, so by 26

$$\theta^l\psi'\left(N^l(n)e^{fb,l}\right) = \left[\frac{\frac{p}{1-(1-p)\delta}(1+\lambda^h)}{\frac{p}{1-(1-p)\delta}(1+\lambda^h) - \lambda^h(\theta^h-\theta^l)(1-p)^n}\right]c'(e^{fb,l}) > c'(e^{fb,l}) \tag{27}$$

Hence, 27 implies $N^l(n) < N^{fb,l}$ and that $N^l(n+1) > N^l(n)$, since ψ is concave and $(1-p)^n$ is decreasing with n. Moreover, 27 also implies that as $n \to \infty$, $N^l(n) \to N^{fb,l}$, since $(1-p)^n \to 0$. In this case, $\pi^l(n+1) > \pi^l(n)$, so by

$$(1-\delta)\sum_{i=n+1}^{\infty}\delta^{i-n}\pi^l(i) > \delta\pi^l(n) = \delta\left[\theta^l\psi\left(N^l(n)e^{fb,l}\right) - N^l(n)\overline{u} - N^l(n)c(e^{fb,l})\right]$$

Now, for $\delta \in (\delta^l, \delta^h(p))$,

$$-(1-\delta)N^{fb,l}c(e^{fb,l}) + \delta\left[\theta^l\psi\left(N^{fb,l}e^{fb,l}\right) - N^{fb,l}\overline{u} - N^{fb,l}c(e^{fb,l})\right] \geq 0$$

and, then, by concavity of π^l with respect to N and since $\psi(0) = 0$, we have

$$-(1-\delta)N^l(n)c(e^{fb,l}) + \delta\left[\theta^l\psi\left(N^l(n)e^{fb,l}\right) - N^l(n)\overline{u} - N^l(n)c(e^{fb,l})\right] \geq 0$$

for all $N^l(n) < N^{fb,l}$, which implies that all $(IC^{f,l})$ constraints hold at the solution found with $\lambda^l(n) = 0$ for all $n \in \mathbb{N}$.

Finally, we show below that the optimal self-enforcing relational plan for $\delta \in (\delta^l_{pi}, \delta^h_{pi}(p))$ found above indeed satisfies the (TT^l) constraint. For that, we will use two properties of the optimal self-enforcing relational plan. The first one is that the (TT^h) constraint holds with equality. This property follows directly from the fact that the $(DE^{f,h})$ constraint also holds with equality at the optimum. Thus, we have

$$\begin{aligned}
-(1-\delta)N^h b^{h,h} + \delta\Pi^h &= -(1-\delta)N^h b^{h,l} + \delta\Pi^{dl}(1) \\
&= -(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) \\
&\quad + \delta(1-\delta)(\theta^h - \theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^l(1+i)e^l(1+i)\right)
\end{aligned}$$

which can also be written as

$$\begin{aligned}
-(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) &= -(1-\delta)N^h b^{h,h} + \delta\Pi^h \\
&\quad - \delta(1-\delta)(\theta^h - \theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^l(1+i)e^l(1+i)\right)
\end{aligned} \tag{28}$$

The second property is that $N^l(n)e^l(n) \leq N^{fb,l}e^{fb,l}$ for all $n \in \mathbb{N}$, which implies $\psi\left(N^l(n)e^l(n)\right) \leq \psi\left(N^{fb,l}e^{fb,l}\right)$ for all $n \in \mathbb{N}$.

By Proposition 3, the firm faces the same decision at every history $\theta^t \in \Theta^t$ in which $\theta_t = \theta^h$. This means, in particular, that whenever the firm privately observes $\theta_{t+1} = \theta^l$, to falsely announce θ^h is profitable if, and only if, it is also profitable to announce θ^h in every period in the future. As a result, we have that $\Pi^{dh} = \theta^l\psi\left(N^h e^h\right) - N^h\overline{u} - N^h b^{h,h}$.

Now, suppose, seeking a contradiction, that (TT^l) is not satisfied, that is,

$$\begin{aligned}
&-(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) < -(1-\delta)N^h b^{h,h} + \delta\Pi^{dh} \\
\Leftrightarrow &-(1-\delta)N^h b^{h,l} + \delta\Pi^l(1) < -(1-\delta)N^h b^{h,h} + \delta\left[\theta^l\psi\left(N^h e^h\right) - N^h\overline{u} - N^h b^{h,h}\right]
\end{aligned} \tag{29}$$

Using equality 28 in the inequality 29 above, we have

$$\begin{aligned}
&-(1-\delta)N^h b^{h,h}+\delta\Pi^h-\delta(1-\delta)(\theta^h-\theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^l(1+i)e^l(1+i)\right)\\
&<-(1-\delta)N^h b^{h,h}+\delta\left[\theta^l\psi\left(N^h e^h\right)-N^h\overline{u}-N^h b^{h,h}\right]\\
\Leftrightarrow&\Pi^h<\left[\theta^l\psi\left(N^h e^h\right)-N^h\overline{u}-N^h b^{h,h}\right]\\
&+(1-\delta)(\theta^h-\theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^l(1+i)e^l(1+i)\right)
\end{aligned} \tag{30}$$

Next, since we have $N^l(n)e^l(n)\leq N^{fb,l}e^{fb,l}$ for all $n\in\mathbb{N}$,

$$\begin{aligned}
&(1-\delta)(\theta^h-\theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^l(1+i)e^l(1+i)\right)\\
&\leq(1-\delta)(\theta^h-\theta^l)\sum_{i=0}^{\infty}((1-p)\delta)^i\psi\left(N^{fb,l}e^{fb,l}\right)\\
&=(\theta^h-\theta^l)\frac{(1-\delta)}{1-(1-p)\delta}\psi\left(N^{fb,l}e^{fb,l}\right)
\end{aligned}$$

which, together with 30, implies

$$\Pi^h<\left[\theta^l\psi\left(N^h e^h\right)-N^h\overline{u}-N^h b^{h,h}\right]+(\theta^h-\theta^l)\frac{(1-\delta)}{1-(1-p)\delta}\psi\left(N^{fb,l}e^{fb,l}\right) \tag{31}$$

Moreover, since $b^{h,h}\geq c(e^h)$, we have

$$\begin{aligned}
&\theta^l\psi\left(N^{fb,l}e^{fb,l}\right)-N^{fb,l}\overline{u}-N^{fb,l}c(e^{fb,l})\\
>&\theta^l\psi\left(N^h e^h\right)-N^h\overline{u}-N^h c(e^h)\\
\geq&\theta^l\psi\left(N^h e^h\right)-N^h\overline{u}-N^h b^{h,h}
\end{aligned} \tag{32}$$

Together with 31, this implies

$$\begin{aligned}
\Pi^h<&\left[\theta^l\psi\left(N^{fb,l}e^{fb,l}\right)-N^{fb,l}\overline{u}-N^{fb,l}c(e^{fb,l})\right]\\
&+(\theta^h-\theta^l)\frac{(1-\delta)}{1-(1-p)\delta}\psi\left(N^{fb,l}e^{fb,l}\right)\\
=&\frac{(1-\delta)}{1-(1-p)\delta}\left[\theta^h\psi\left(N^{fb,l}e^{fb,l}\right)-N^{fb,l}\overline{u}-N^{fb,l}c(e^{fb,l})\right]\\
&+\frac{p\delta}{1-(1-p)\delta}\left[\theta^l\psi\left(N^{fb,l}e^{fb,l}\right)-N^{fb,l}\overline{u}-N^{fb,l}c(e^{fb,l})\right]
\end{aligned} \tag{33}$$

Consider, then, an alternative relational plan $\tilde{\mathcal{P}} = \left(\tilde{N}_t(\theta^t), \tilde{e}_t(\theta^t), \tilde{w}_t(\theta^t), \tilde{b}_t^h(\theta^t), \tilde{b}_t^l(\theta^t)\right)_{t\geq 0}$ in which $\left(\tilde{N}_t(\theta^t), \tilde{e}_t(\theta^t)\right) = (N^{fb,l}, e^{fb,l})$, $\tilde{w}_t(\theta^t) = \overline{u}$, and $\tilde{b}_t^h(\theta^t) = \tilde{b}_t^l(\theta^t) = c(e^{fb,l})$ for every t and θ^t. Note that $\tilde{\mathcal{P}}$ generates ex-ante expected profits equal to

$$\frac{(1-\delta)}{1-(1-p)\delta}\left[\theta^h \psi\left(N^{fb,l}e^{fb,l}\right) - N^{fb,l}\overline{u} - N^{fb,l}c(e^{fb,l})\right] + \frac{p\delta}{1-(1-p)\delta}\left[\theta^l \psi\left(N^{fb,l}e^{fb,l}\right) - N^{fb,l}\overline{u} - N^{fb,l}c(e^{fb,l})\right]$$

which, by 33, is strictly greater than Π^h. Therefore, if we show $\tilde{\mathcal{P}}$ to be self-enforcing, then we have a contradiction with the fact that the original relational plan is optimal. Indeed, by definition $\tilde{\mathcal{P}}$ satisfies the (IC^w) constraint. Since, for $\delta \geq \delta_{pi}^l$,

$$-(1-\delta)N^{fb,l}c(e^{fb,l}) + \delta\pi^{fb,l} \geq 0$$

and since

$$\frac{(1-\delta)}{1-(1-p)\delta}\left[\theta^h \psi\left(N^{fb,l}e^{fb,l}\right) - N^{fb,l}\overline{u} - N^{fb,l}c(e^{fb,l})\right] + \frac{p\delta}{1-(1-p)\delta}\left[\theta^l \psi\left(N^{fb,l}e^{fb,l}\right) - N^{fb,l}\overline{u} - N^{fb,l}c(e^{fb,l})\right] > \pi^{fb,l}$$

$\tilde{\mathcal{P}}$ satisfies both the $(IC^{f,h})$ and $(IC^{f,l})$ constraints. Moreover, (TT^h) and (TT^l) are clearly satisfied with equality, since the action profiles and contracts are the same for both productivities. Thus, $\tilde{\mathcal{P}}$ is self-enforcing and we have a contradiction.

Therefore, it must be that the (TT^l) constraint is satisfied in the original optimal relational plan. □

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