Term Structure Movements Implicit in Asian Option Prices *

Caio Almeida
Graduate School of Economics
Getulio Vargas Foundation
Praia de Botafogo 190,
11th Floor, RJ, 5521-2559-5827
calmeida@fgv.br

José Vicente
Research Department
Central Bank of Brazil
Av. Pres. Vargas 730,
7th Floor, RJ, 5521-2189-5083
jose.valentim@bcb.gov.br

January 30, 2008

Abstract

Motivated by the usefulness of interest rate asian options as hedging instruments, we pioneer implementations of dynamic term structure models that adopt those options in the estimation process. The models are used to analyze pricing and hedging implications of term structure dynamic movements once options are included (or not) in the estimation process. Based on analytical formulas for bonds and asian options under a multi-factor Gaussian model, we analyze how options affect the shape, risk premium and hedging structure of the dynamic factors. Results indicate that the inclusion of options has strong effects on the *loadings* of the slope and curvature factors, and considerably affects the risk premium and hedging structure of all dynamic factors. As a robustness check to the Gaussian model, a dynamic term structure model with stochastic volatility ($A_1(3)$) is also implemented. The $A_1(3)$ partially confirms results obtained with the Gaussian model.

Keywords: Dynamic Term Structure Models, Latent Factors, Bond Risk Premium, Interest Rates Asian Option Pricing.
JEL Code: C51, G12

*We thank useful comments from seminar participants at Catholic University of Rio de Janeiro, Federal University of Santa Catarina, Ibmec Business School SP, the Sixth Brazilian Meeting of Finance, and XI School of Time Series and Econometrics. The views expressed are those of the authors and do not necessarily reflect the views of the Central Bank of Brazil. The first author gratefully acknowledges financial support given by CNPq-Brazil. Any remaining errors are our responsibility alone.

1 Introduction

Interest rate asian options are securities depending on the accumulated/average value of the short-term rate, which are extremely useful hedging instruments for corporations with periodic cash-flows. They are popular over-the-counter instruments cheaper than their vanilla counter-parts (caps, floors), less subjective to price manipulation, and offering simpler hedging strategies than regular interest rate options (see Longstaff (1995); Chacko and Das (2002)). Nevertheless, despite the existence of innumerous theoretical results on the pricing of interest rates asian options, previous research has been limited to cross-section option pricing[1].

In fact, an analysis of the risk premium structure of interest rate asian options will demand an efficient use of computationally intensive Fourier transform methods[2] to price those options within a dynamic model. In this case, optimizing over the model parameter vector is a much more demanding task than pricing the option on a cross section, when the Fourier inverse transform can be easily performed via a numerical integration procedure.

In this paper, we fill the gap between theory and practice by estimating multi-factor dynamic term structure models with joint data on bonds and interest rate asian options. Our contribution lies in carefully implementing these models to analyze how options affect the shape, risk premium, and hedging structure of the corresponding dynamic term structure factors. This is done by implementing different versions of dynamic models, either including only bonds on the estimation process, or including bonds and options.

We implement two versions of a three-factor Gaussian model, and as a robustness check to the hypothesis of constant volatility, we also implement two versions of a three-factor affine model with one factor driving stochastic

[1]Geman and Yor (1993) use the theory of Bessel processes to price asian options under a Cox et al. (CIR, 1985) process. Longstaff (1995) adopt the Vasicek (1977) model to analyze properties of options on average interest rates. Leblanc and Scaillet (1998) present theoretical results on the pricing of interest rate asian options under the Vasicek (1977) and CIR (1985) models, based on Laplace transforms. Cheuk and Vorst (1999) adopt a Hull and White (1990) model. Dassios and Nagaradjasarma (2003) also adopt the CIR (1985) model, and use a series expansion representation methodology to price the asian option. Work adopting multi-factor models includes Bakshi and Madan (1999) who use a two-factor CIR (1985) model and Fourier transform methods, and Chacko and Das (2002) who also adopt Fourier transform methods to price those options under general affine term structure models.

[2]See Bakshi and Madan (2000), Duffie et al. (2000), and Chacko and Das (2002).

volatility ($A_1(3)$; Dai and Singleton (2000)). Closed-form formulas for bonds and asian option prices allow for an efficient implementation of the Gaussian model. In contrast, the $A_1(3)$ model is implemented via an adaptation of the Edgeworth expansion method proposed by Collin Dufresne and Goldstein (2002b) to price swaptions. Results for the $A_1(3)$ model qualitatively confirm the results obtained with the Gaussian model in what concerns the shape of term structure movements, and partially confirm results concerning the risk premium structure.

The two versions of each model (Gaussian, $A_1(3)$) are estimated respectively by Maximum Likelihood under the Gaussian model and Quasi Maximum Likelihood under the $A_1(3)$ model: For each model, the first version adopts only bonds data (the bond version), and the other combines bonds and at-the-money fixed-maturity options data (the option version). Empirical results indicate that options affect basically three dimensions of the dynamic model: *Loadings* of term structure movements, bond premium decomposition, and dynamic first order hedging terms[3].

Empirical results show that the level is a robust factor common to both versions of each estimated model, while slope and curvature are less persistent under the option version of each model (see Figures 3 and 4). These movements present much higher mean reversion rates under the option version, indicating that while information contained in bonds and at-the-money options agree on the main factor driving term structure movements, the information implicit in those option prices suggest faster variations for the secondary movements of the term structure.

Under the Gaussian model, bond risk premia is slightly less volatile on the option version, and is more concentrated on the level factor. Under the $A_1(3)$ model, the level factor has a small contribution to the term structure of risk premiums under both versions, while the slope factor appears as the main factor driving premium. Although each model presents its own risk premium decomposition, both agree that when switching from the bond version to the option version of each model, level increases its participation on the risk premium decomposition while curvature decreases its participation.

A comparison of the two estimated versions of the Gaussian model further

[3]In order to keep an adequate size and a better organization of the paper, under the $A_1(3)$ model, we only report results related to shape and risk premium structure of dynamic factors, but not related to dynamic hedging. It should be clear however that once the model was implemented, the hedging analysis could be naturally pursued without incurring in higher computational costs.

reveals that the bond version better captures the term structure of bond yields, but is out-performed by the option version on the option pricing and hedging exercises. From a hedging perspective, the bond version is only able to capture 5.10% price movements of the at-the-money option adopted, contrasted to a 94.74% fraction for the option version[4]. When analyzing the dynamic hedging weights attributed to each factor under each version, it is clear that both versions give no importance to the curvature dynamic factor when hedging the at-the-money option, while level and slope weights are much more volatile under the option version of the model.

Related works include the papers by Umantsev (2001), Bikbov and Chernov (2004), Li and Zhao (2006), Joslin (2007), and Almeida and Vicente (2007). Bikbov and Chernov (2004) adopt a joint dataset of Eurodollar bonds and options to economically discriminate among affine models with different volatility specifications. While they test how including options affects the shape of term structure factors, they do not present a risk premium analysis nor any dynamic hedging analysis. Umantsev (2001) estimates three-factor affine dynamic term structure models simultaneously adopting swaps and swaption prices to analyze the risk premium structure of both swap-based and joint swap/swaption-based models. Like Bikbov and Chernov however, he does not provide an analysis of hedging performance of each version of those models. In contrast, Li and Zhao (2006) implement Quadratic Term Structure Models (Ahn et al. (2002)) to test their hedging performance with respect to cap derivatives. Note however that their dynamic models are estimated based on *only bonds data*, while caps are only considered as separate instruments to test hedging performance. In contrast, we explicitly include options in our estimation process. Joslin (2007) implements four-factor affine models with a flexible covariance structure that allows to simultaneously price bonds and bond options (swaptions). He analyzes the hedging implications of such models finding that dynamic hedging strategies using bonds alone produce reasonable good hedges for derivative positions. While he focuses on the dynamic hedging properties of his models based on *swaptions data*, we look at different aspects (including hedging) of how *interest rate asian options* affect term structure movements.

It should be clear that the main innovation of our work is to provide an

[4]Note that this was expected since the option version is perfectly pricing this option, and the 4.79% variability of prices not captured in the delta-hedge is due to second order effects. The only reason to provide hedging results under the option version is to allow comparison of dynamic hedging weights across versions.

empirical analysis of term structure movements based on dynamic models estimated *with asian options data*, which appear to be of interest by their own. In this sense, there is only one work which is close in spirit to the present paper: Almeida and Vicente (2007), who implement a dynamic term structure model with bonds and asian options to analyze the volatility risk premium structure of these joint markets. The dynamic term structure model implemented belongs to the class of unspanned stochastic volatility models (USV; Collin Dufresne and Goldstein (2002a)), generating an incomplete bond market. For this specific reason, the only way to estimate their proposed USV model is with the use of *joint bonds and options data*. In clear contrast, in the present paper we analyze dynamic term structure models generating complete bond markets, and are able to estimate different versions of each dynamic model, some including only bonds data, while others include both bonds and options data. This ability to estimate different versions is fundamental here since we are interested in contrasting the distinct versions of each dynamic model with respect to how they affect term structure movements.

In summary, from a theoretical viewpoint, we provide efficient ways of implementing multi-factor affine term structure models (Gaussian and $A_1(3)$) including interest rate asian options in the estimation. From an empirical sense, our contribution is to provide a careful examination of how including asian options in the estimation process of those dynamic models will affect the loadings, the risk premium structure, and first order hedging of term structure movements. Our results should be useful for risk management and portfolio management purposes, as well as also work as a tool for practitioners to quickly price bonds and asian options with analytical formulas.

The paper is organized as follows. Section 2 describes the market of ID-futures (bonds), and IDI interest rate asian options. Section 3 presents the model, the pricing of zero-coupon bonds and IDI options, and first order dynamic hedging properties of such options. Section 4 describes and implements the estimation process under each version. Section 5 compares the two dynamic versions of the Gaussian model considering the empirical dimensions described above. Results on the $A_1(3)$ model in what concerns the shape and risk premium structure of term structure movements are also presented. Section 6 concludes. Appendix A contains theoretical results on the pricing of fixed income instruments under the Gaussian model. Appendix B show how we price bonds and interest rate asian options under the $A_1(3)$ model. Appendix C presents a detailed description of the Maximum Likelihood estimation procedure adopted.

2 Data and Market Description

2.1 ID-Futures

The One-Day Inter Bank Deposit Future Contract (ID-Future) with maturity T is a future contract whose underlying asset is the accumulated daily ID rate[5] capitalized between the trading time t ($t \leq T$) and T. The contract size corresponds to R$ 100,000.00 (one hundred thousand Brazilian Reals) discounted by the accumulated rate negotiated between the buyer and the seller of the contract.

This contract is very similar to a zero coupon bond, except that it pays margin adjustments every day. Each daily cash flow is the difference between the settlement price[6] on the current day and the settlement price on the day before corrected by the ID rate of the day before.

The Brazilian Mercantile and Futures Exchange (BM&F) is the entity that offers the ID-Future. The number of authorized contract-maturity months is fixed by BM&F (on average, there are about twenty authorized contract-maturity months for each day but only around ten are liquid). Contract-maturity months are the first four months subsequent to the month in which a trade has been made and, after that, the months that initiate each following quarter. Expiration date is the first business day of the contract-maturity month.

2.2 ID Index and its Option Market

The ID index (IDI) is defined as the accumulated ID rate. If we associate the continuously-compounded ID rate to the short term rate r_t then

$$IDI_t = IDI_0 \cdot \mathrm{e}^{\int_0^t r_u \mathrm{d}u}. \tag{1}$$

This index, computed on every workday by BM&F, has been fixed to the value of 100000 points in January 2, 1997, and has actually been resettled to its initial value a couple of times, most recently in January 2, 2003.

[5]The ID rate is the average one-day inter bank borrowing/lending rate, calculated by CETIP (Central of Custody and Financial Settlement of Securities) every workday. The ID rate is expressed in effective rate per annum, based on 252 business-days.

[6]The settlement price at time t of an ID-Future with maturity T is equal to R$ 100,000.00 discounted by its closing price quotation.

An IDI option with time of maturity T is an European option where the underlying asset is the IDI and whose payoff depends on IDI_T. When the strike is K, the payoff of an IDI option is $L_c(T) = (IDI_T - K)^+$ for a call and $L_p(T) = (K - IDI_T)^+$ for a put.

As can be noticed, IDI options have a peculiar characteristic which is not shared by usual fixed income international options: They are asian options. Their payoff depends on the integral of the short-term rate through the path between the trading date t and the option maturity date T. This makes them specially suited to complement the theoretical papers on interest rate asian options that have been discussed in Section 1.

BM&F is also the entity that provides the IDI call options. Strike prices (expressed in index points) and the number of authorized contract-maturity months are established by BM&F. Contract-maturity months can happen to be any month, and the expiration date is the first business day of the maturity month. Usually, there are 30 authorized series within each day, from which about a third are liquid.

2.3 Data

Data consists on time series of ID-futures yields for all different liquid maturities, and prices of IDI options for different strikes and maturities, covering the period from January, 2003 to December, 2005.

BM&F maintains a daily historical database with prices and number of trades for all ID-futures and IDI options that have been traded within a day. Interest rates for zero coupon bonds with fixed maturities are estimated with a cubic interpolation scheme applied to the ID-futures dataset. On the estimation process of the Gaussian model, yields from bonds with fixed maturities of 1, 21, 63, 126, 189, 252 and 378 days are adopted[7].

Regarding options, two different databases are selected. The first, used on the estimation of the option version of the dynamic models, is composed by an at-the-money fixed-maturity IDI call[8], with time to maturity equal to 95 days[9]. The second is composed by picking up within each day the most

[7]There exist deals within this market with longer maturities (up to ten years) but the liquidity is considerably lower.

[8]Moneyness is defined by the ratio of the present value of strike over current IDI value.

[9]The at-the-money IDI call prices are obtained by an interpolation of Black implied volatilities in a similar procedure to that adopted to construct original VIX volatilities.

liquid IDI call[10].

The first database containing options is used to estimate the dynamic models (option versions), and the second is used to test the pricing performance of the two versions. As hedging can not be tested with the database on the most liquid IDI options because moneyness and/or maturity change through time, the hedging is performed using the at-the-money options of the first database[11].

After excluding weekends, holidays, and no-trade workdays, there exists a total of 748 daily observations of yields from zero coupon bonds, and option prices[12].

3 The Model

The uncertainty in the economy is characterized by a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$. The existence of a pricing measure $\mathbb{Q}$ under which discounted bond prices are martingales is assumed, and the model is specified through the definition of the short term rate r_t as a sum of N Gaussian random variables:

$$r_t = \phi_0 + \sum_{i=1}^{N} X_t^i, \tag{2}$$

where the dynamics of process X is given by

$$\mathrm{d}X_t = -\kappa X_t \mathrm{d}t + \rho \mathrm{d}W_t^{\mathbb{Q}}, \tag{3}$$

with $W^{\mathbb{Q}}$ being an N-dimensional brownian motion under $\mathbb{Q}$, κ a diagonal matrix with κ_i in the i_{th} diagonal position, and ρ is a matrix responsible for correlation among the X factors. The connection between martingale probability measure $\mathbb{Q}$ and objective probability measure $\mathbb{P}$ is given by Girsanov's

[10] Moneyness and time-to-maturity of liquid options are readily available upon request.

[11] In this case it should be clear that the option version will outperform the bond version since the first perfectly prices the at-the-money option. However, as explained in the empirical section the most interesting aspect of this hedging exercise is to compare the dynamic allocations provided to each term structure movement by each model.

[12] This sample size is compatible with that found in other recent academic studies containing derivatives data from emerging economies (see for instance, Pan and Singleton (2007)). In addition, as our study contains high frequency data, the number of observations (748) adopted to estimate the dynamic term structure model is large enough to avoid small-sample biases.

Theorem with an essentially affine (Duffee (2002))[13] market price of risk

$$\mathrm{d}W_t^{\mathbb{P}} = \mathrm{d}W_t^{\mathbb{Q}} - \lambda_X X_t \mathrm{d}t, \tag{4}$$

where λ_X is an $N \times N$ matrix and $W^{\mathbb{P}}$ is a brownian motion under $\mathbb{P}$.

Lemma 1 *Let $y(t,T) = \int_t^T r_u du$. Then, under measure $\mathbb{Q}$ and conditional on the sigma field $\mathcal{F}_t$, y is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, where*

$$M(t,T) = \phi_0 \tau + \sum_{i=1}^{N} \frac{1 - \mathrm{e}^{-\kappa_i \tau}}{\kappa_i} X_t^i \tag{5}$$

and

$$\begin{aligned} V(t,T) = {} & \sum_{i=1}^{N} \tfrac{1}{\kappa_i^2} \left(\tau + \tfrac{2}{\kappa_i}\mathrm{e}^{-\kappa_i \tau} - \tfrac{1}{2\kappa_i}\mathrm{e}^{-2\kappa_i \tau} - \tfrac{3}{2\kappa_i} \right) \sum_{j=1}^{N} \rho_{ij}^2 + \\ & +2 \sum_{i=1}^{N} \sum_{k>i} \tfrac{1}{\kappa_i \kappa_k} \left(\tau + \tfrac{\mathrm{e}^{-\kappa_i \tau}-1}{\kappa_i} + \tfrac{\mathrm{e}^{-\kappa_k \tau}-1}{\kappa_k} - \tfrac{\mathrm{e}^{-(\kappa_i+\kappa_k)\tau}-1}{\kappa_i+\kappa_k} \right) \sum_{j=1}^{N} \rho_{ij}\rho_{kj}, \end{aligned} \tag{6}$$

where $\tau = T - t$.

Proof. See Appendix A. ■

3.1 Pricing Zero Coupon Bonds

Let $P(t,T)$ denote the time t price of a zero coupon bond maturing at time T, paying one monetary unit. It is known that Multi-factor Gaussian models offer closed-form formulas for zero coupon bond prices. The next lemma presents a simple proof of this fact for the particular model in hand.

Lemma 2 *The price at time t of a zero coupon bond maturing at time T is*

$$P(t,T) = \mathrm{e}^{A(\tau) + B(\tau)' X_t}, \tag{7}$$

where $A(\tau) = -\phi_0 \tau + \frac{1}{2} V(t,T)$ and $B(\tau)$ is a column vector with $-\frac{1-\mathrm{e}^{-\kappa_i \tau}}{\kappa_i}$ as the i_{th} element.

[13]Constrained for admissibility purposes (see Dai and Singleton (2000)).

Proof. See Appendix A. ■

Using Equation (7) and Itô's lemma we can obtain the dynamics of bond prices under the martingale measure $\mathbb{Q}$

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t \mathrm{d}t + B(\tau)'\rho \mathrm{d}W_t^{\mathbb{Q}}. \tag{8}$$

To hold this bond, investors will ask for an instantaneous expected excess return. Then, under the objective measure, the bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + z^i(t,T))\mathrm{d}t + B(\tau)'\rho \mathrm{d}W_t^{\mathbb{P}}. \tag{9}$$

Applying Girsanov's Theorem to change measures the instantaneous premium is obtained as

$$z^i(t,T) = B(\tau)'\rho\lambda_X X_t. \tag{10}$$

3.2 Pricing Interest Rate Asian Options

IDI options are continuous-time interest rate asian options. Theoretical results and cross-section pricing of interest rate asian options can be found in Geman and Yor (1993) (for the CIR model), Longstaff (1995) (for the Vasicek (1977) model), Leblanc and Scaillet (1998) (Vasicek and CIR models), Cheuk and Vorst (1999) (Hull and White (1990) model), Bakshi and Madan (1999) (two-factor CIR model), Chacko and Das (2002) (general affine models), and Dassios and Nagaradjasarma (2003) (CIR model). Each of these papers build on different techniques including Fourier/Laplace transforms, representations in series of functions, and Bessel processes theory. In this section, we propose analytical formulas for asian option prices that allow for an efficient implementation of the dynamic term structure model, complementing thus the above mentioned theoretical work on an empirical dimension.

Following Duffie and Kan (1996) and Dai and Singleton (2000), we adopt multiple factors to drive the uncertainty of the yield curve. Multiple factors driving term structure movements have been advocated since the work of Litterman and Scheinkman (1991), and recently have been adopted in most fixed income empirical applications (see Singleton (2006)). Option pricing is provided in what follows.

Denote by $c(t,T)$ the time t price of a call option on the IDI index, with time of maturity T, and strike price K, then

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_u \mathrm{d}u} \max(IDI_T - K, 0)|\mathcal{F}_t\right] = \\ = \mathrm{E}^{\mathbb{Q}}\left[\max(IDI_t - K\mathrm{e}^{-y(t,T)}, 0)|\mathcal{F}_t\right]. \tag{11}$$

Lemma 3 *The price at time t of the above mentioned option is*

$$c(t,T) = IDI_t\Phi(d) - KP(t,T)\Phi(d - \sqrt{V(t,T)}), \tag{12}$$

where Φ denotes the cumulative normal distribution function, and d is given by

$$d = \frac{\log \frac{IDI_t}{K} - \log P(t,T) + V(t,T)/2}{\sqrt{V(t,T)}}. \tag{13}$$

Proof. See Appendix A. ■

If $p(t,T)$ is the price at time t of the IDI put with strike K and maturity T then, by the put-call parity

$$p(t,T) = KP(t,T)\Phi(\sqrt{V(t,T)} - d) - IDI_t\Phi(-d). \tag{14}$$

3.3 Hedging IDI Options

When hedging an instrument, we are interested in the composition of a portfolio which approximately neutralizes variations on the price of this instrument. To that end, we should make use of a set of additional instruments which present dynamics related to the dynamics of the targeted instrument. Alternatively, it is known that each state variable driving uncertainty on the term structure is responsible for one type of movement. These movements are represented by the state variables loadings as a function of time to maturity (see Section 5 for a concrete example). Similarly to Li and Zhao (2006), this research assumes that those state variables are tradable assets which can be used as instruments to compose the hedging portfolio. The main advantage of this approach is to avoid introduction of additional sources of error due to approximate relations between the hedging instruments and the state variables.

The goal of this hedging analysis is to identify if the bond version of model captures the dynamics of IDI options. A delta hedging procedure is performed by equating the first derivatives (with respect to state variables) of the hedging portfolio to the first derivatives (with respect to state variables) of the instrument being hedged, which was chosen, for illustration purposes, to be one contract of a call on the IDI index with strike K, and time of maturity T. Letting Π_t denote the time t value of the hedging portfolio, by assumption it must satisfy

$$\Pi_t = q_t^1 X_t^1 + q_t^2 X_t^2 + ... + q_t^N X_t^N, \tag{15}$$

where q_t^i is the number of units of X_t^i in the hedging portfolio, and X_t^i is the i^{th} term structure dynamic factor. By simply equating the first order variation of Π_t to the first order variation of the IDI option price $c(t,T)$, it is obtained that $q_t^i = \frac{\partial c(t,T)}{\partial X_t^i}$. Calculating the partial derivatives using Equation (12) it follows that

$$q_t^i = \frac{1 - \mathrm{e}^{-\kappa_i \tau}}{\kappa_i \sqrt{V(t,T)}} \Big[IDI_t \Phi'(d) + KP(t,T)\sqrt{V(t,T)}\Phi(d - \sqrt{V(t,T)}) - KP(t,T)\Phi'(d - \sqrt{V(t,T)}) \Big]. \tag{16}$$

On the empirical exercise presented bellow, Equation (16) is used to readjust the hedging on a daily basis.

4 Parameters Estimation

In this section, two versions of a three factor Gaussian model are estimated[14]. Model parameters are obtained based on a maximum likelihood procedure adopted by Chen and Scott (1993) and described in Appendix C, in an extended form considering options in the estimation process:

- On the bond version, only ID-futures data, in form of fixed maturity zero coupon bond implied yields, is used in the estimation process. Bonds with maturities of 1, 126, and 252 days are observed without error[15]. For each fixed t, the state vector is obtained through the

[14]According to a principal component analysis applied to the covariance matrix of observed yields, three factors are sufficient to describe 99.5% of the variability of the term structure of ID bonds.

[15]Inversions of the state vector considering other combinations of bonds were also tested offering similar qualitative results in what regards parameter estimation and bond pricing errors.

solution of the following linear system:

$$
\begin{aligned}
rb_t(0.00397) &= -\tfrac{A(0.00397,\phi)}{0.00397} - \tfrac{B(0.00397,\phi)'}{0.00397}X_t \\
rb_t(0.5) &= -\tfrac{A(0.5,\phi)}{0.5} - \tfrac{B(0.5,\phi)'}{0.5}X_t \\
rb_t(1) &= -\tfrac{A(1,\phi)}{1} - \tfrac{B(1,\phi)'}{1}X_t.
\end{aligned} \tag{17}
$$

where rb_t represents the vector of ID-yields observed at time t.

Bonds with time to maturity of 21, 63, 189 and 378 days, are assumed to be observed with gaussian errors u_t uncorrelated in the time dimension:

$$
\begin{aligned}
rb_t(0.0833) &= -\tfrac{A(0.0833,\phi)}{0.0833} - \tfrac{B(0.0833,\phi)'}{0.0833}X_t + u_t(0.0833) \\
rb_t(0.25) &= -\tfrac{A(0.25,\phi)}{0.25} - \tfrac{B(0.25,\phi)'}{0.25}X_t + u_t(0.25) \\
rb_t(0.75) &= -\tfrac{A(0.75,\phi)}{0.75} - \tfrac{B(0.75,\phi)'}{0.75}X_t + u_t(0.75) \\
rb_t(1.5) &= -\tfrac{A(1.5,\phi)}{1.5} - \tfrac{B(1.5,\phi)'}{1.5}X_t + u_t(1.5).
\end{aligned} \tag{18}
$$

The Jacobian matrix is

$$
Jac_t = \begin{bmatrix} -\tfrac{B(0.00397,\phi)'}{0.00397} \\ -\tfrac{B(0.5,\phi)'}{0.5} \\ -\tfrac{B(1,\phi)'}{1} \end{bmatrix}; \tag{19}
$$

- On the option version, options are included in the estimation procedure. This is done by assuming that the instruments observed without error are bonds with maturities of 1 and 189 days, and the at-the-money IDI call option with time to maturity of 95 days, whose time t observed price is denoted by cs_t. The state vector is obtained through the solution of the following non-linear system

$$
\begin{aligned}
rb_t(0.00397) &= -\tfrac{A(0.00397,\phi)}{0.00397} - \tfrac{B(0.00397,\phi)'}{0.00397}X_t \\
rb_t(0.75) &= -\tfrac{A(0.75,\phi)}{0.75} - \tfrac{B(0.75,\phi)'}{0.75}X_t \\
cs_t &= c(t, t+0.377),
\end{aligned} \tag{20}
$$

where $c(t,T)$ is given by Equation (11).

Bonds with time to maturity equal to 21, 63, 252, and 378 days, are priced with uncorrelated gaussian errors u_t:

$$
\begin{aligned}
rb_t(0.0833) &= -\tfrac{A(0.0833,\phi)}{0.0833} - \tfrac{B(0.0833,\phi)'}{0.0833} X_t + u_t(0.0833) \\
rb_t(0.25) &= -\tfrac{A(0.25,\phi)}{0.25} - \tfrac{B(0.25,\phi)'}{0.25} X_t + u_t(0.25) \\
rb_t(1) &= -\tfrac{A(1,\phi)}{1} - \tfrac{B(1,\phi)'}{1} X_t + u_t(1) \\
rb_t(1.5) &= -\tfrac{A(1.5,\phi)}{1.5} - \tfrac{B(1.5,\phi)'}{1.5} X_t + u_t(1.5).
\end{aligned}
\tag{21}
$$

The Jacobian matrix is

$$
Jac_t = \begin{bmatrix} -\tfrac{B(0.00397,\phi)'}{0.00397} \\ \\ -\tfrac{B(0.75,\phi)'}{0.75} \\ \\ q_t \end{bmatrix},
$$

where $q_t = \left[q_t^1, \ldots, q_t^N\right]$ with q_t^i calculated for $T = t + 0.377$.

Under both versions of the model, the transition probability $p(X_t|X_{t-1};\phi)$ is a three-dimensional gaussian distribution with known mean and variance as functions of parameters appearing in ϕ.

Tables 1 and 2 present respectively the values of the parameters estimated for each version of the model. Standard deviations are obtained by the BHHH method (see Davidson and MacKinnon (1993)). Under both versions most of the parameters are significant at a 95% confidence interval, except for a few risk premia parameters, and one parameter which comes from the correlation matrix of the brownian motions. The long term short rate mean ϕ_0 was fixed equal to 0.18, compatible with the ID short-rate sample mean of 0.1778[16].

Appendix B describes how we proceed to price bonds and asian options under the $A_1(3)$ model that we estimated. Appendix C describes how the $A_1(3)$ model was estimated (based on Quasi Maximum Likelihood).

[16]Optimization including this parameter was also experimented, but generated results with higher standard errors for a considerable fraction of the parameter vector.

5 Empirical Results

Figure 1 presents the evolution of bond yields extracted from ID-futures data, from January, 2003 to December, 2005. Yields range from a maximum of 25% observed in the beginning of the sample period to a minimum of 15% in February, 2004. Figure 2 presents the average observed and model implied term structures of interest rates for zero coupon bonds, under each estimated version of the Gaussian model. Its clear from the picture that on the pricing of bonds, the bond version outperforms the option version [17]. Under the bond version, the mean absolute error for yields of zero coupon bonds with time to maturity 21, 63, 189 and 378 days are respectively 18.10 bps[18], 6.93 bps, 1.76 bps and 11.52 bps. The errors standard deviations, which provide a metric for their time series variability, are 24.52 bps, 9.52 bps, 2.26 bps and 14.07 bps. Under the option version, the mean absolute error for yields of bonds with time to maturity 21, 63, 126 and 378 days are respectively 29.72 bps, 14.89 bps, 12.93 bps and 39.03 bps, with standard deviations of 35.37 bps, 17.70 bps, 15.92 bps and 46.54 bps.

The $A_1(3)$ model presents very similar pricing errors to the Gaussian model, under both versions. Under the bond version, the mean absolute error for yields of zero coupon bonds with time to maturity 21, 63, 189 and 378 days are respectively 18.34 bps, 7.21 bps, 1.86 bps and 11.75 bps, with values for all maturities approximately 0.2 bps above the Gaussian model. The standard deviations of these errors are respectively 24.58 bps, 9.62 bps, 2.29 bps and 14.07 bps, practically equal to their Gaussian counterpart values. Under the option version of the $A_1(3)$ model, the mean absolute error for yields of zero coupon bonds with time to maturity 21, 63, 189 and 378 days are respectively 25.84 bps, 25.21 bps, 11.90 bps and 36.89 bps, and error standard deviations are respectively 36.02 bps, 36.20 bps, 15.95 bps, and 46.38 bps. Note that the Gaussian and $A_1(3)$ models differ only on the error of the 63-day yield under their option version[19].

[17] Under the bond version, the three dimensional latent vector X, characterizing uncertainty in the economy, is fully inverted from bonds data. In contrast, the option version only captures the yields of two bonds without errors, because the third instrument priced without error is an at-the-money option.

[18] Bps stands for basis points. One basis point is equivalent to 0.01%.

[19] Probably because the option 95-day maturity might be affecting the CIR process driving volatility in the $A_1(3)$ model in an asymmetric way (when compared to the Gaussian model), slightly distorting the bond pricing of bonds with maturities close to the option.

5.1 Term Structure Movements and Bond Risk Premium

Figure 3 presents the loadings of the three dynamic factors under each version of the Gaussian model (solid lines correspond to the bond version, dotted lines to the option version). The level factor[20] presents loadings indistinguishable across versions. However, slope and curvature factors are clearly different. They both present higher curvatures under the option version, suggesting that option investors tend to react faster (than bond investors) to news that affect the term structure of bond risk premiums in an asymmetric way[21]. Similarly, Figure 4 presents the loadings of the three dynamic factors under each version of the $A_1(3)$ model. Note the similarity between Figures 3 and 4, indicating that the modification in the shape of term structure dynamic factors once asian options are included in the estimation process affect the Gaussian model and the $A_1(3)$ model in very related ways. It happens that under the $A_1(3)$ model the mean reversion rate of the curvature factor is slightly less affected once options are included. In fact, while under the Gaussian model, the mean reversion rate of the curvature factor increases from 6.34 (bond version) to 37.63 (option version), under the $A_1(3)$ model, it increases from 6.84 to 15.87. Figure 5 presents the state variables driving each term structure movement, for the two versions of the Gaussian model[22]. Note that the time series of the slope and curvature factors, under the option version, present spikes that are consistent with fast mean reverting variables[23].

An important point related to the modification of term structure movements is to understand what are the implications on investor's interpretation of risks when options are or not included in the estimation process. This might be addressed in at least two ways: By observing the time series of model implied bond risk premiums and contrasting across versions, or directly observing bond risk premium decomposition as a combination of term structure movements, under each version.

[20] It is the one with slowest mean reversion speed and responsible for explaining most of the variation on yields.

[21] Note that a shock on the level factor affects the risk premium term structure in a symmetric way.

[22] The average value of the short-rate (ϕ_0) should be added to the level state variable, in order to obtain the level factor.

[23] Results for the time series of dynamic factor are very similar under the $A_1(3)$ model, and are readily available upon request.

Figure 6 presents pictures of the term structures of bond instantaneous risk premium (measured by Equation (10)) in different instants of time, for the Gaussian model. Note that the cross section of premiums is very distinct across versions, and in particular, the longer the maturity the larger the difference between the risk premium implied by each version. In addition, under the option version, the term structure of risk premiums is better approximated by a linear function, and risk premiums are in general lower. The time series behavior of the premiums might be better observed in Figure 7, which presents the evolution of the instantaneous risk premium for the 1-year bond, under the two versions. During the period from September of 2003 to December of 2004, the premium is significantly higher under the bond version. That was a period where interest rates were consistently being lowered by the Central Bank of Brazil, and in this context, the smaller premium (under the option version) indicates the possibility of an inertia of bond investors in re-estimating their expectations for long term behavior of interest rates, as opposed to a fast reaction of option market players.

The risk premium decomposition across movements of the term structure provides a direct way of identifying the shifts in importance of factors once options are adopted in the estimation process. From Equation (10), it is clear that risk premium is a linear combination of the state variables: $z(t, t+\tau) = a_1(\tau)X_t^1 + a_2(\tau)X_t^2 + a_3(\tau)X_t^3$. Figure 8 presents, for the Gaussian model, the term structure of risk premiums decomposed for each maturity among the three movements: Level, slope and curvature. Solid lines represent the bond version and dashed lines the option version. For each fixed maturity, the sum of the absolute weights on the three movements gives 100%. The decomposition presents a clearly distinct pattern for maturities bellow and above 0.5 years, under both versions. For instance, under the bond version, the curvature factor explains more than 70% of the premium for short maturities while curvature and slope together explain the premium for longer maturities. Under the option version the level factor explains most of the premium for longer maturities while it splits this role with the curvature factor for shorter maturities. Under both versions the slope contributes negatively to the risk premium decomposition. In general, risk premium is more sensitive to the curvature and slope factors under the bond version, and to the level and curvature factors under the option version. Contrasting factor loadings and risk premiums, it is possible to identify that the use of options data provides less persistent slope and curvature movements, but prices the most persistent factor (level). On the other hand, when only bonds are adopted in the estimation process, secondary movements (slope and curvature) are more persistent, but are priced in stead of the level movement (still the most persistent factor). Results tend to suggest that within the Brazilian fixed income market, option investors are more concerned with monetary policy through the level of interest rates, while bond investors are more concerned with the volatility of interest rates through curvature and slope (see Litterman et al. (1991)).

The corresponding risk premium decomposition under the $A_1(3)$ model is presented in Figure 9. Similarly to the Gaussian model, once options are included in the estimation the importance of the level factor in the risk premium decomposition increases, while the importance of the curvature factor decreases. However, the two models disagree on how the price the slope factor. While under the Gaussian model the slope decreases its participation in risk premium once options are included, under the $A_1(3)$ model the opposite appears to happen. This might be the consequence of having stochastic volatility under the $A_1(3)$ model driven by the level factor, generating a tension between first and second conditional moments previously observed in the literature of affine dynamic models (see Dufee (2002) and Duarte (2004)). In any case, we are not advocating that the changes when asian options are included in the estimation process should be robust to changes in the dynamic term structure model chosen. The interesting point is that both Gaussian and $A_1(3)$ models appear to agree in enough points so as to allow a fixed income manager to safely consider implementing the simpler Gaussian model (instead of $A_1(3)$) to extract information on shapes and risk premium structure of term structure movements in joint bond / interest rate asian option markets.

5.2 Pricing and Hedging Options

The goal of the next exercise is to understand how useful could be the inclusion of options on the estimation process of the dynamic model when pricing and hedging options. Since under the option version, an at-the-money option is used to invert the state vector, this exercise is only interesting if out-of-sample options are adopted. For this reason, the database of most liquid IDI call options is adopted when comparing pricing performances across versions.

Figure 10 presents observed option prices versus model implied prices. Points represent the bond version and x's the option version, under the Gaussian model. For modeling purposes, an ideal relation would be a 45 degree line passing through the origin with angular coefficient equal to 1 (solid line in Figure 10). Under the bond version, a linear regression of observed prices depending on model prices, presents a $R^2 = 97.5\%$, an angular coefficient equal to 1.0423 (p-value < 0.01) and a linear coefficient of 86.83 (p-value < 0.01). The high R^2 indicates that the option prices obtained under the bond version correctly captures the time series variability of observed option prices (high correlation). However, the high value for the linear coefficient implies that the bond version consistently underestimates option prices. The underestimation of option prices is confirmed by Figure 11, which presents the relative error defined by model price minus observed price, divided by observed price. Note how under the bond version it is smaller than zero during most of the time. The absolute relative pricing error presents an average of 17.53%[24].

When the same regression is provided for the option version, the R^2 is slightly bellow, achieving 97.2%, probably due to some mispricing of options with prices in the range $[1500, 3000]$ (see Figure 10). On the other hand, both the angular coefficient of 1.0121 (p-value < 0.01) and the linear coefficient of 11.67 (p-value $= 0.14$) are closer to ideal values. The smaller linear coefficient indicates that once options are adopted in the estimation process they help the dynamic model to better capture the level of option prices. The dotted line in Figure 11 presents the relative pricing error for the option version. Note that it clearly outperforms the bond version, except for the end of the sample period when it overestimates option prices. It achieves an average absolute value of 10.75%, a 40% improvement with respect to the bond version.

The next step implements a dynamic delta-hedging strategy on the fixed-maturity at-the-money IDI call option[25]. Note that if the hedging is effective, variations on the hedging portfolio should approximately offset variations on the option price. The correlation coefficients between these variations are 5.10% and 94.74% for the bond and option versions respectively, directly suggesting that the option based version is much more efficient when hedging. In fact, one could expect with no surprises that the option version would be able to perform an excellent hedging since the at-the-money option is

[24]For comparison purposes, see Jagannathan et al. (2003) who price U.S. caps adopting a three-factor CIR model estimated with U.S Libor and swaps data.

[25]On the hedging analysis a fixed-maturity, fixed-moneyness option is adopted, otherwise changes in prices would reflect not only the price dynamics but also changes on the type of the option.

inverted to extract the state vector. In this sense, the hedging error for the option model is essentially a second order error not captured by the delta-hedging procedure. However, the result of interest is the comparison of dynamic hedging weights across versions. Figure 12 displays the number of units in the hedging portfolio invested on each state variable. Observe that in both versions of the model the option is more sensitive to the level factor and less sensitive to the curvature factor, and in particular under the option version, the allocations to both level and slope factors are much more volatile. This high volatility of the allocations reflects the fact that at-the-money options are highly sensitive to changes in their underlying assets, which in the particular case are interest rates.

6 Conclusion

Asian options are important over-the-counter instruments that are very valuable hedging resources. Targeting to understand how these options affect corresponding bond markets, we use the technology of affine processes (Duffie and Kan (1996)) to implement different dynamic models including asian options in the estimation process. In particular, with the use of analytical formulas for bonds and asian options, two versions of a dynamic multi-factor Gaussian model were estimated, one adopting only bonds data (bond version), the other combining bonds and interest rate asian options data (option version). The main interest was to verify if and how asian options change the loadings, risk premium, and hedging structures of dynamic term structure factors.

We identified that interest rate asian options bring information that primarily affect the speeds of mean reversion of the slope and curvature of the yield curve, and that strongly affect the decomposition of bond risk premia. In addition, when delta-hedging an at-the-money option, both implemented versions of the Gaussian model give small importance to the curvature factor, while the option version presents much more volatile weights on slope and level factors, which seem to be necessary to capture the dynamics of option prices.

Based on Edgeworth expansion techniques and forward probability methods, a model generating stochastic volatility for interest rates ($A_1(3)$) was also implemented to verify the validity of results obtained with the Gaussian model. The $A_1(3)$ obtains very similar results (to the Gaussian model) in what regards the effects of asian options on the loadings of term structure movements, and partially confirms the results on the way options change bond risk premium structure.

Our results complement theoretical studies on the pricing of interest rate asian options, because we pioneer implementations of dynamic models that include these options in the estimation process. They should be useful for portfolio and risk management purposes as simple and effective tools for pricing and hedging fixed income instruments with the use of interest rate asian options.

Appendix A

Proof. Lemma 1
By Ito's rule, for each $t < T$ the unique strong solution of (3) is[26]

$$X_T^i = X_t^i \mathrm{e}^{-\kappa_i (T-t)} + \sum_{j=1}^{N} \rho_{ij} \int_t^T \mathrm{e}^{-\kappa_i (T-s)} \mathrm{d}W_s^j, \quad i = 1, \ldots, N.$$

Then

$$r_T = \phi_0 + \sum_{i=1}^{N} \left(X_t^i \mathrm{e}^{-\kappa_i (T-t)} + \sum_{j=1}^{N} \rho_{ij} \int_t^T \mathrm{e}^{-\kappa_i (T-s)} \mathrm{d}W_s^j \right).$$

Stochastic integration by parts implies that

$$\int_t^T X_u^i \mathrm{d}u = \int_t^T (T-u)\, \mathrm{d}X_u^i + (T-t)\, X_t^i. \tag{22}$$

By definition of X, the integral in the right-hand side can be written as

$$\int_t^T (T-u)\, \mathrm{d}X_u^i = -\kappa_i \int_t^T (T-u)\, X_u^i \mathrm{d}u + \sum_{j=1}^{N} \rho_{ij} \int_t^T (T-u)\, \mathrm{d}W_u^j.$$

Note also that

$$\int_t^T (T-u)\, X_u^i \mathrm{d}u =$$

$$= X_t^i \int_t^T (T-u)\, \mathrm{e}^{-\kappa_i (u-t)} \mathrm{d}u + \sum_{j=1}^{N} \rho_{ij} \int_t^T (T-u) \int_t^u \mathrm{e}^{-\kappa_i (u-s)} \mathrm{d}W_s^j \mathrm{d}u.$$

Calculating separately the last two integrals, the following result holds

$$\int_t^T (T-u)\, \mathrm{e}^{-\kappa_i (u-t)} \mathrm{d}u = \left(\frac{T-t}{\kappa_i} + \frac{\mathrm{e}^{-\kappa_i (u-t)} - 1}{\kappa_i^2} \right)$$

[26]In this appendix we drop the superscript $\mathbb{Q}$ and denote the N-dimensional brownian motion $W^{\mathbb{Q}}$ simply by W.

and, again by integration by parts,

$$\int_t^T (T-u) \int_t^u \mathrm{e}^{-\kappa_i(u-s)} \mathrm{d}W_s^j \mathrm{d}u =$$

$$= \int_t^T \left(\int_t^u \mathrm{e}^{\kappa_i s} \mathrm{d}W_s^j\right) \mathrm{d}_u \left(\int_t^u (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v\right) =$$

$$= \left(\int_t^T \mathrm{e}^{\kappa_i u} \mathrm{d}W_u^j\right) \left(\int_t^T (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v\right) -$$

$$- \int_t^T \left(\int_t^u (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v\right) \mathrm{e}^{\kappa_i u} \mathrm{d}W_u^j =$$

$$= \int_t^T \left(\int_u^T (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v\right) \mathrm{e}^{\kappa_i u} \mathrm{d}W_u^j =$$

$$\tfrac{1}{\kappa_i} \int_t^T \left(T-u+\tfrac{\mathrm{e}^{-\kappa_i(T-u)}-1}{\kappa_i}\right) \mathrm{d}W_u^j.$$

Substituting the previous terms in Equation (22), the following result holds

$$\int_t^T X_u^i \mathrm{d}u = (T-t)\, X_t^i -$$

$$-\kappa_i \left[X_t^i \left(\tfrac{T-t}{\kappa_i} + \tfrac{\mathrm{e}^{-\kappa_i(T-t)}-1}{\kappa_i^2}\right) + \sum_{j=1}^N \tfrac{\rho_{ij}}{\kappa_i} \int_t^T \left(T-u+\tfrac{\mathrm{e}^{-\kappa_i(T-u)}-1}{\kappa_i}\right) \mathrm{d}W_u^j\right] +$$

$$+ \sum_{j=1}^N \rho_{ij} \int_t^T (T-u)\, \mathrm{d}W_u^j =$$

$$= -\tfrac{\mathrm{e}^{-\kappa_i(T-u)}-1}{\kappa_i} X_t^i + \sum_{j=1}^N \rho_{ij} \int_t^T -\tfrac{\mathrm{e}^{-\kappa_i(T-u)}-1}{\kappa_i} \mathrm{d}W_u^j =$$

$$= \tfrac{1-\mathrm{e}^{-\kappa_i(T-u)}}{\kappa_i} X_t^i + \tfrac{1}{\kappa_i} \sum_{j=1}^N \rho_{ij} \int_t^T \left(1-\mathrm{e}^{-\kappa_i(T-u)}\right) \mathrm{d}W_u^j,$$

that is,

$$\int_t^T X_u^i \mathrm{d}u = \frac{1-\mathrm{e}^{-\kappa_i(T-u)}}{\kappa_i} X_t^i + \frac{1}{\kappa_i} \sum_{j=1}^N \rho_{ij} \int_t^T \left(1-\mathrm{e}^{-\kappa_i(T-u)}\right) \mathrm{d}W_u^j. \quad (23)$$

Then $y(t,T) = \phi_0 (T-t) + \sum_{i=1}^N \int_t^T X_u^i \mathrm{d}u$ conditional on $\mathcal{F}_t$ is normally distributed (see Duffie (2001)) with mean

$$M(t,T) = \phi_0 (T-t) + \sum_{i=1}^N \frac{1-\mathrm{e}^{-\kappa_i(T-t)}}{\kappa_i} X_t^i, \quad (24)$$

where the fact that the stochastic integral in (23) is a martingale was used. The variance of $y(t,T)|\mathcal{F}_t$ is

$$V\left(t,T\right)=\mathrm{var}^{\mathbb{Q}}\left[\sum_{i=1}^{N}\frac{Y_i}{\kappa_i}|\mathcal{F}_t\right], \tag{25}$$

where $Y_i=\sum_{j=1}^{N}\rho_{ij}\int_t^T\left(1-\mathrm{e}^{-\kappa_i(T-u)}\right)\mathrm{d}W_u^j$. Then

$$V\left(t,T\right)=\sum_{i=1}^{N}\frac{\mathrm{var}^{\mathbb{Q}}\left(Y_i|\mathcal{F}_t\right)}{\kappa_i^2}+2\sum_{i=1}^{N}\sum_{k>i}\frac{\mathrm{cov}^{\mathbb{Q}}\left(Y_i,Y_k|\mathcal{F}_t\right)}{\kappa_i\kappa_k}.$$

Using Ito's isometry

$$V\left(t,T\right)=\sum_{i=1}^{N}\tfrac{1}{\kappa_i^2}\sum_{j=1}^{N}\rho_{ij}^2\int_t^T\left(1-\mathrm{e}^{-\kappa_i(T-u)}\right)^2\mathrm{d}u+$$
$$+2\sum_{i=1}^{N}\sum_{k>i}\tfrac{1}{\kappa_i\kappa_k}\sum_{j=1}^{N}\rho_{ij}\rho_{kj}\int_t^T\left(1-\mathrm{e}^{-\kappa_i(T-u)}\right)\left(1-\mathrm{e}^{-\kappa_k(T-u)}\right)\mathrm{d}u. \tag{26}$$

At this point, simple integration produces

$$V\left(t,T\right)=\sum_{i=1}^{N}\tfrac{1}{\kappa_i^2}\left(\tau+\tfrac{2}{\kappa_i}\mathrm{e}^{-\kappa_i\tau}-\tfrac{1}{2\kappa_i}\mathrm{e}^{-2\kappa_i\tau}-\tfrac{3}{2\kappa_i}\right)\sum_{j=1}^{N}\rho_{ij}^2+$$
$$+2\sum_{i=1}^{N}\sum_{k>i}\tfrac{1}{\kappa_i\kappa_k}\left(\tau+\tfrac{\mathrm{e}^{-\kappa_i\tau}-1}{\kappa_i}+\tfrac{\mathrm{e}^{-\kappa_k\tau}-1}{\kappa_k}-\tfrac{\mathrm{e}^{-(\kappa_i+\kappa_k)\tau}-1}{\kappa_i+\kappa_k}\right)\sum_{j=1}^{N}\rho_{ij}\rho_{kj}, \tag{27}$$

where $\tau=T-t$. ■

Proof. Lemma 2
The martingale condition for bond prices (Duffie (2001)) gives:

$$P(t,T)=\mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_u\mathrm{d}u}|\mathcal{F}_t\right]=\mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-y(t,T)}|\mathcal{F}_t\right]. \tag{28}$$

Now the normality of variable $y(t,T)|\mathcal{F}_t$ (Lemma 1), and a simple property of the mean of log-normal distributions complete the proof. ■

Proof. Lemma 3
By Equation (11) the proof consists of a simple calculation of the expectation

$\mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-y}, 0\right)|\mathcal{F}_t\right]$.

$$c\left(t,T\right) = \mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-y}, 0\right)|\mathcal{F}_t\right] =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V(t,T)}} \max\left(IDI_t - K\mathrm{e}^{-y}, 0\right) \mathrm{e}^{-\frac{(y-M(t,T))^2}{2V(t,T)}} \mathrm{d}y = \tag{29}$$

$$= \int_{\log(K/IDI_t)}^{\infty} \frac{1}{\sqrt{2\pi V(t,T)}} \left(IDI_t - K\mathrm{e}^{-y}\right) \mathrm{e}^{-\frac{(y-M(t,T))^2}{2V(t,T)}} \mathrm{d}y.$$

Making the substitution $z = \frac{y-M(t,T)}{\sqrt{V(t,T)}}$ the following result holds:

$$c\left(t,T\right) = \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} \left(IDI_t - K\mathrm{e}^{-z\sqrt{V(t,T)}-M(t,T)}\right) \mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z =$$

$$= IDT_t \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z - K\int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z\sqrt{V(t,T)}-M(t,T)-\frac{1}{2}z^2}\mathrm{d}z = \tag{30}$$

$$= IDI_t\Phi\left(d\right) - K\mathrm{e}^{-M(t,T)+\frac{V(t,T)}{2}} \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(z+\sqrt{V(t,T)}\right)^2}\mathrm{d}z.$$

where d is given by Equation (13). Making a new substitution $v = z + \sqrt{V\left(t,T\right)}$ and using Lemma 2 results in Equation (12). ■

Appendix B

6.1 The Stochastic Volatility Model ($A_1(3)$)

We implement a version of an $A_1(3)$ model as a robustness check for the results obtained with the Gaussian model. An $A_1(3)$ model is characterized by the presence of three state variables with one of the them driving the conditional volatility of the short term rate (Dai and Singleton (2000)). Our $A_1(3)$ model is a particular form of the most general one, where the short term rate is the sum of one independent CIR process (see Cox et al (1985)) with two Gaussian processes:

$$r_t = \phi_0 + X_t + Y_t + Z_t, \tag{31}$$

with:

$$\mathrm{d}X_t = \kappa_X(\theta - X_t)dt + \rho_X\sqrt{X_t}\mathrm{d}W_X^{\mathbb{Q}}(t), \tag{32}$$

$$\mathrm{d}YZ_t = \begin{bmatrix} \mathrm{d}Y_t \\ \mathrm{d}Z_t \end{bmatrix} = -\begin{bmatrix} \eta_Y & 0 \\ 0 & \eta_Z \end{bmatrix}\begin{bmatrix} Y_t \\ Z_t \end{bmatrix}\mathrm{d}t + \begin{bmatrix} \rho_Y & 0 \\ \rho_{YZ} & \rho_Z \end{bmatrix}\begin{bmatrix} \mathrm{d}W_Y^{\mathbb{Q}}(t) \\ \mathrm{d}W_Z^{\mathbb{Q}}(t) \end{bmatrix}$$

$$= \eta YZ_t\mathrm{d}t + \rho\mathrm{d}W_{YZ}^{\mathbb{Q}}(t), \tag{33}$$

where $W_X^{\mathbb{Q}}$, $W_Y^{\mathbb{Q}}$ and $W_Z^{\mathbb{Q}}$ are independent brownian motions under the pricing measure $\mathbb{Q}$, and where we use the following short notation:

$$\mathrm{d}YZ_t = \begin{bmatrix} \mathrm{d}Y_t \\ \mathrm{d}Z_t \end{bmatrix} \quad \text{and} \quad \mathrm{d}W_{YZ}^{\mathbb{Q}}(t) = \begin{bmatrix} \mathrm{d}W_Y^{\mathbb{Q}}(t) \\ \mathrm{d}W_Z^{\mathbb{Q}}(t) \end{bmatrix}.$$

The transition from the pricing measure $\mathbb{Q}$ to the objective probability measure $\mathbb{P}$ is given by the extended affine market prices of risks by Cheridito et al. (2006)[27]:

$$\mathrm{d}W_X^{\mathbb{Q}}(t) = \mathrm{d}W_X^{\mathbb{P}}(t) + \frac{1}{\sqrt{X_t}}\left(\lambda_0^X + \lambda_1^X X_t\right)\mathrm{d}t \tag{34}$$

[27] Under the Gaussian model, this market price of risk coincides with the essentially affine market price by Duffee (2002), indicating that both models (Gaussian and $A_1(3)$) are implemented with the most general market prices that maintain affine dynamics under both risk neutral and objective measures.

and

$$dW_{YZ}^{\mathbb{Q}}(t) = dW_{YZ}^{\mathbb{P}}(t) + \left(\lambda_0^{YZ} + \lambda_1^{YZ} YZ_t\right) dt, \tag{35}$$

where λ_0^X and λ_1^X are real numbers, λ_0^{YZ} is a vector in $\mathbb{R}^2$, and λ_1^{YZ} is a 2×2 matrix.

By the independence of the CIR process, it follows directly from the results of Section 3 and from Brigo and Mercurio (2001) that the time t price of a zero coupon bond maturing at time T is given by ($\tau = T - t$):

$$P(t,T) = e^{A(\tau)+B_X(\tau)X_t+B_Y(\tau)Y_t+B_Z(\tau)Z_t}, \tag{36}$$

where

$$A(\tau) = -\phi_0\tau + \frac{2\kappa_X\theta}{\rho_X}\ln\left(\frac{2\gamma e^{\frac{\kappa_X+\gamma}{2}}}{2\gamma + (\kappa_X+\gamma)(e^{\tau\gamma}-1)}\right) + \frac{V_{YZ}(t,T)}{2},$$

$$B_X(\tau) = -\frac{2(e^{\tau\gamma}-1)}{2\gamma + (\kappa_X+\gamma)(e^{\tau\gamma}-1)},$$

$$B_Y(\tau) = -\frac{1-e^{\eta_Y\tau}}{\eta_Y},$$

and

$$B_Z(\tau) = -\frac{1-e^{\eta_Z\tau}}{\eta_Z},$$

with $\gamma = \sqrt{\kappa_X^2 + 2\rho_X^2}$ and V_{YZ} given by Equation (6), with $N = 2$ and $\kappa = \eta$.

According to the explanation in Section 3, the pricing of IDI options demands knowledge of the distribution of $y(t,T)$ conditioned on the information available at time t. If r_t is a Gaussian process then $y|\mathcal{F}_t$ is normally distributed. However, in the $A_1(3)$ case there is not a simple numerical procedure to calculate probabilities related to y. Nevertheless, the moment of order m of $e^{-y(t,T)}$ can be calculated, since it is the price of a bond when the short term rate is $m \times r_t$. Therefore we can use an Edgeworth expansion technique to obtain IDI asian option prices (see Collin-Dufresne and Goldstein (2002b) for its use on swaptions pricing). In the next lines we describe how to apply this technique in our $A_1(3)$ version. Using the forward measure approach (Geman et al. (1995)), the price of the IDI option is given by:

$$c(t,T) = IDI_t\mathcal{P}^{\mathbb{Q}}(e^{-y} < IDI_t/K) - KP(t,T)\mathcal{P}^T(e^{-y} < IDI_t/K),$$

where $\mathcal{P}$ denotes probabilities and the superscript T represents the forward measure.

The probabilities in the right-hand side of the above equation can be calculated using the Edgeworth expansion. The Edgeworth expansion is basically an expansion of a distribution around the normal distribution. Following Collin-Dufresne and Goldstein (2002b) we expand up to the seventh order. In this case, the probabilities are equal to $1 - \sum_{j=0}^{7} \alpha_j \beta_j$, where the coefficients α_j are ratios of polynomials of order 7 in the moments of $\mathrm{e}^{-y(t,T)}$ and β_j are simple functions of the cumulative normal function and the first two moments of $\mathrm{e}^{-y(t,T)}$[28]. The moments of $\mathrm{e}^{-y(t,T)}$ under both the risk neutral measure $\mathbb{Q}$ and the forward measure are obtained as described in the previous paragraph. However, the computation of these moments under the forward measure are slightly more difficult, since r_t under forward measure T follows a Hull and White (1990) process with time-varying parameters. Therefore we do not have closed-form expressions for bonds, and adopt the Runge-Kutta method to solve numerically the coupled pair of differential equations satisfied by terms A and B appearing in the bond expression.

[28] Collin-Dufresne and Goldstein (2002b) provide the precise expressions for the coefficients α_j and β_j.

Appendix C - Estimation Procedure

Estimation Under the Gaussian model

In this work, under the Gaussian model, the maximum likelihood estimation procedure described in Chen and Scott (1993), is extended to deal with options[29]. The following bond yields are observed along H different days: $rb_t(1/252)$, $rb_t(21/252)$, $rb_t(63/252)$, $rb_t(126/252)$, $rb_t(189/252)$, $rb_t(1)$ and $rb_t(1.5)$[30]. Let rb represent the $H \times 7$ matrix containing the yields for all H days. In addition, the price cs_t for an at-the-money call with time to maturity 95/252 years is observed during the same H days. Let cs be the vector of length H that represents these call prices. The ID bonds and the at-the-money IDI call are called reference market instruments. Denote by $rmi = [rb, cs]$ the $H \times 8$ matrix containing the yields and the price of these reference market instruments. Assume that model parameters are represented by vector ϕ and a time unit equal to Δt. Finally, let $g_i(X_t; t, \phi)$ be the function that maps reference market instrument i into state variables.

As three factors are adopted to estimate the model, it is assumed that reference market instruments, say i_1, i_2 and i_3, are observed without error. For each fixed t, the state vector is obtained through the solution of the following system:

$$
\begin{aligned}
g_{i_1}(X_t; t, \phi) &= rmi(t, i_1) \\
g_{i_2}(X_t; t, \phi) &= rmi(t, i_2) \\
g_{i_3}(X_t; t, \phi) &= rmi(t, i_3).
\end{aligned}
\tag{37}
$$

Reference market instruments i_4, i_5, i_6, i_7 and i_8, are assumed to be observed with gaussian uncorrelated errors u_t:

$$
\begin{aligned}
& rmi(t, \begin{bmatrix} i_4 & i_5 & i_6 & i_7 & i_8 \end{bmatrix}) - u_t = \\
& \begin{bmatrix} g_{i_4}(X_t; t, \phi) & g_{i_5}(X_t; t, \phi) & g_{i_6}(X_t; t, \phi) & g_{i_7}(X_t; t, \phi) & g_{i_8}(X_t; t, \phi) \end{bmatrix}
\end{aligned}
\tag{38}
$$

[29]For the estimation of more general dynamic term structure models based on joint bond-option data, see for instance, Umantsev (2001), Han (2007), or Almeida et al. (2006), among others.

[30]$rb_t(\tau)$ stands for the time t yield of a bond with time to maturity τ.

The log-likelihood function can be written as

$$L(\phi, rb) = \sum_{t=2}^{H} \log p(X_t|X_{t-1};\phi) - \\ - \sum_{t=2}^{H} \log|Jac_t| - \frac{H-1}{2}\log|\Omega| - \frac{1}{2}\sum_{t=2}^{H} u_t'\Omega^{-1}u_t, \tag{39}$$

where:

1. $Jac_t = \begin{bmatrix} \frac{\partial g_{i_1}(X_t;t,\phi)}{\partial X_t} \\ \frac{\partial g_{i_2}(X_t;t,\phi)}{\partial X_t} \\ \frac{\partial g_{i_3}(X_t;t,\phi)}{\partial X_t} \end{bmatrix}$ is the Jacobian matrix of the transformation defined by Equation (37);

2. Ω represents the covariance matrix for u_t, estimated using the sample covariance matrix of the u_t's implied by the extracted state vector;

3. $p(X_t|X_{t-1};\phi)$ is the transition probability from X_{t-1} to X_t under the objective probability measure $\mathbb{P}$.

The final objective within this procedure is to estimate vector ϕ which maximizes function $L(\phi, rb)$. In order to avoid possible local minima, several different starting parameter vectors are tested and, for each one, a search for the optimal point is performed, using Nelder-Mead Simplex algorithm for non-linear optimization and gradient-based optimization methods.

Estimation Under the $A_1(3)$ model

Under the $A_1(3)$ model, we also adopt a procedure similar to Chen and Scott (1993) but in stead of Maximum Likelihood (ML), a Quasi Maximum Likelihood estimation is performed. Although the transition probabilities are known to be under the particular $A_1(3)$ model adopted the product of a non-central chi-square density and a Gaussian density, for stability purposes we decided to implement QML since we have analytical formulas for the first two conditional moments of the state vector under any affine model. Other than applying QML, the rest of the estimation procedure is precisely as described above for the Gaussian model.

References

[1] Ahn D.H., R. Dittmar, and R. Gallant (2002). Quadratic Term Structure Models: Theory and Evidence. *Review of Financial Studies*, **15**, 1, 243-288.

[2] Almeida C., J. Graveline, and S. Joslin (2006). Do Fixed Income Options Contain Information About Excess Returns?, Working Paper, Stanford Graduate School of Business.

[3] Almeida C. and J. Vicente (2007). Identifying Volatility Risk Premia from Fixed Income Asian Options. Working Paper, Graduate School of Economics, Getulio Vargas Foundation.

[4] Bakshi G. and D. Madan (1999). A Unified Treatment of Average-Rate Contingent Claims with Applications. Working Paper, University of Maryland.

[5] Bakshi G. and D. Madan (2000). Spanning and Derivative-Security Valuation. *Journal of Financial Economics*, **55**, 2, 205-238.

[6] Bikbov R. and M. Chernov (2005). Term Structure and Volatility: Lessons from the Eurodollar Markets. Working Paper, Division of Finance and Economics, Columbia University.

[7] Brigo D. and F. Mercurio (2001). *Interest Rate Models: Theory and Practice*, Springer Verlag.

[8] Chacko G. and S. Das (2002). Pricing Interest Rate Derivatives: A General Approach. *Review of Financial Studies*, **15**, 1, 195-241.

[9] Chen R.R. and L. Scott (1993). Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates. *Journal of Fixed Income*, **3**, 14-31.

[10] Cheridito P., D. Filipovic, and R. Kimmel (2007). Market Price of Risk Specifications for Affine Models: Theory and Evidence. *Journal of Financial Economics*, **83**, 1, 123-170.

[11] Cheuk T.H.F. and T.C.F. Vorst (1999). Average Interest Rate Caps. *Computational Economics*, **14**, 183-196.

[12] Collin Dufresne P., and R.S. Goldstein (2002a). Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility. *Journal of Finance*, **LVII**, 4, 1685-1730.

[13] Collin Dufresne P. and R.S. Goldstein (2002b). Pricing Swaptions Within an Affine Framework. *Journal of Derivatives*, **Vol. 10**, 1.

[14] Cox J. C., J.E. Ingersoll and S.A. Ross (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, **53**, 385-407.

[15] Dai Q. and K. Singleton (2000). Specification Analysis of Affine Term Structure Models. *Journal of Finance*, **LV**, 5, 1943-1977.

[16] Dai Q. and K. Singleton (2002). Expectation Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure. *Journal of Financial Economics*, **63**, 415-441.

[17] Dassios A. and J. Nagaradjasarma (2003). Pricing Asian Options On Interest Rates in the CIR Model. Working Paper, Department of Statistics, London School of Economics.

[18] Davidson R. and J.G. MacKinnon (1993). *Estimation and Inference in Econometrics*. Oxford University Press.

[19] Duarte J. (2004). Evaluating an Alternative Risk Preference in Affine Term Strcuture Models. *Review fo Financial Studies*, **17**, 2, 370-404.

[20] Duffee G. R. (2002). Term Premia and Interest Rates Forecasts in Affine Models. *Journal of Finance*, **57**, 405-443.

[21] Duffie D. (2001). *Dynamic Asset Pricing Theory*, Princeton University Press.

[22] Duffie D. and R. Kan (1996). A Yield Factor Model of Interest Rates. *Mathematical Finance*, Vol. 6, **4**, 379-406.

[23] Duffie D., J. Pan, and K. Singleton (2000). Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica* **68** 1343-1376.

[24] Hull J. And A. White (1990). Pricing Interest Rate Derivatives Securities. *Review of Financial Studies*, **3**, 4, 573-592.

[25] Geman H., N. El Karoui, and J.-C. Rochet (1995). Changes of Numeraire, Changes of Probability Measure and Option Pricing. *Journal of Applied Probability*, **32**, 2, 443-458.

[26] Geman H. and M. Yor (1993). Bessel Processes, Asian Options and Perpetuities. *Mathematical Finance*, **3**, 349-375.

[27] Han B. (2007). Stochastic Volatility and Correlations of Bond Yields. *Journal of Finance*, **vol LXII**, 3, 1491-1523.

[28] Jagannathan R., A. Kaplin, and S. Sun (2003). An Evaluation of Multifactor CIR Models using LIBOR, Swap Rates, and Cap and Swaptions Prices. *Journal of Econometrics*, **116**, 113-146.

[29] Joslin S. (2007). Pricing and Hedging Volatility Risk in Fixed Income Markets. Working Paper, Sloan School of Management, MIT.

[30] Leblanc B. and O. Scaillet (1998). Path Dependent Options on Yields in the Affine Term Structure Model. *Finance and Stochastics*, **2**, 349-367.

[31] Li H. and F. Zhao (2006). Unspanned Stochastic Volatility: Evidence from Hedging Interest Rates Derivatives. *Journal of Finance*, **61**, 1, 341-378.

[32] Litterman R. and J.A. Scheinkman (1991). Common Factors Affecting Bond Returns. *Journal of Fixed Income*, **1**, 54-61.

[33] Litterman R., J.A. Scheinkman, and L. Weiss (1991). Volatility and the Yield Curve. *Journal of Fixed Income*, **1**, 49-53.

[34] Longstaff F.A. (1995). Hedging Interest Rate Risk with Options on Average Interest Rates. *Journal of Fixed Income*, March, 37-45.

[35] Pan J. and K. Singleton (2007). Default and Recovery Implicit in the Term Structure of Sovereign CDS Spreads. forthcoming at *Journal of Finance*.

[36] Singleton K. (2006). *Empirical Dynamic Asset Pricing*, Princeton University Press.

[37] Umantsev L. (2001). Econometric Analysis of European Libor-Based Options within Affine Term Structure Models. PhD Thesis, Department of Management Science and Engineering, Stanford University.

[38] Vasicek O.A. (1977). An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, **5**, 177-188.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Error}}$
κ_1	6.3435	0.0889	**71.34**
κ_2	1.6082	0.0174	**92.47**
κ_3	0.0003	0.00001	**12.65**
ρ_{11}	0.0919	0.0021	**43.07**
ρ_{21}	-0.0216	0.0034	**6.30**
ρ_{22}	0.0400	0.0010	**40.22**
ρ_{31}	-0.0008	0.0016	0.47
ρ_{32}	-0.0192	0.0004	**50.85**
ρ_{33}	0.0112	0.0001	**108.49**
$\lambda_X(11)$	-329.7170	109.0627	**3.02**
$\lambda_X(21)$	42.9899	68.3982	0.62
$\lambda_X(22)$	0.5462	12.0799	0.05
$\lambda_X(31)$	-200.4261	39.4736	**5.07**
$\lambda_X(32)$	258.7188	10.6457	**24.30**
$\lambda_X(33)$	-75.3815	7.9478	**9.48**
ϕ_0	0.18	-	-

Table 1: Parameters and Standard Errors Obtained Under the Bond Version for the Gaussian Model.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Error}}$
κ_1	37.6296	10.8910	**3.46**
κ_2	3.4565	0.1858	**18.60**
κ_3	0.0003	0.00002	**16.96**
ρ_{11}	0.0919	0.0040	**23.96**
ρ_{21}	-0.0415	0.0044	**9.41**
ρ_{22}	0.0729	0.0016	**45.45**
ρ_{31}	-0.0006	0.0017	0.39
ρ_{32}	-0.0332	0.0016	**20.72**
ρ_{33}	0.0194	0.0003	**69.66**
$\lambda_X(11)$	-240.0116	129.1894	1.86
$\lambda_X(21)$	-137.1462	63.9335	**2.15**
$\lambda_X(22)$	0.0376	12.4838	0.00
$\lambda_X(31)$	-260.0849	84.7153	**3.07**
$\lambda_X(32)$	16.917	26.6624	0.63
$\lambda_X(33)$	-278.9916	13.1735	**21.17**
ϕ_0	0.18	-	-

Table 2: Parameters and Standard Errors Obtained Under the Option Version for the Gaussian Model.

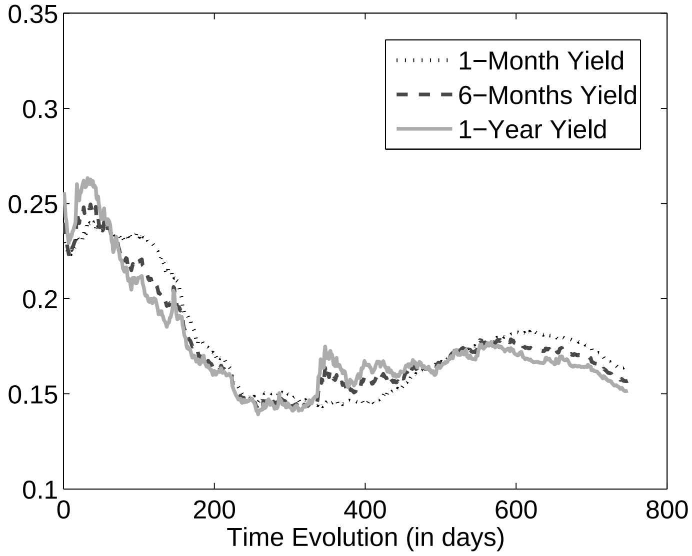

Figure 1: Time Series of Brazilian Bonds Yields: From January, 2003 to December, 2005.

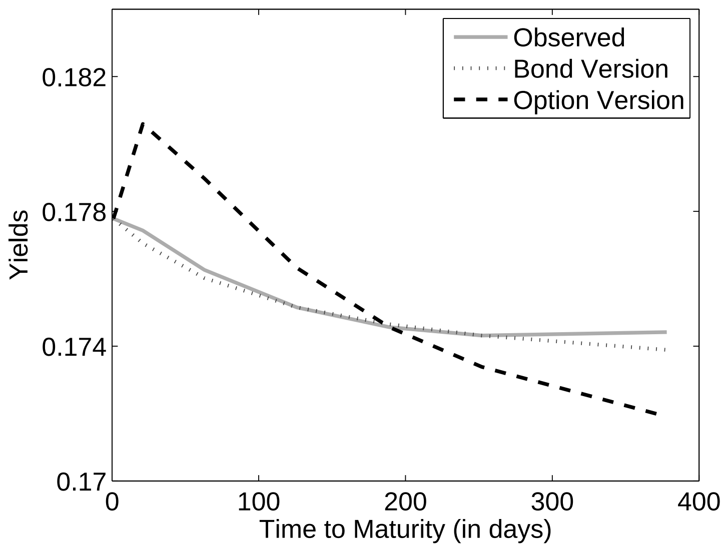

Figure 2: Average Observed and Model-Implied Cross Section of Yields Under the Gaussian Model.

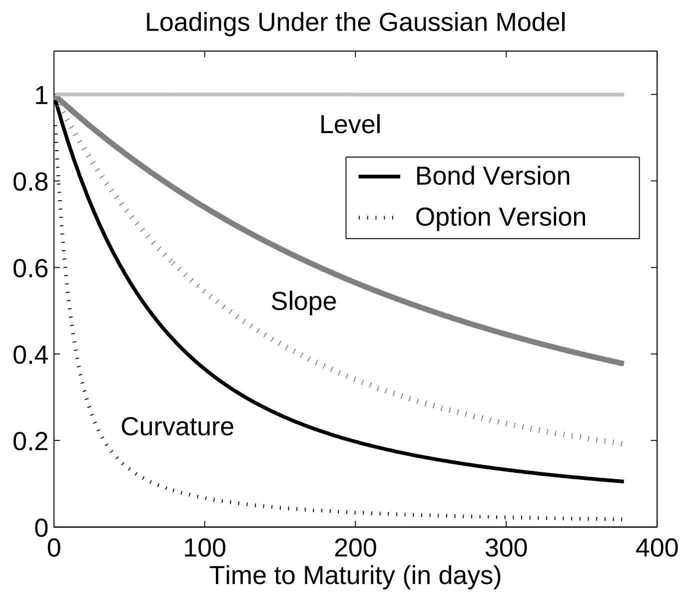

Figure 3: Loadings of the Three Dynamic Factors Under the Gaussian Model.

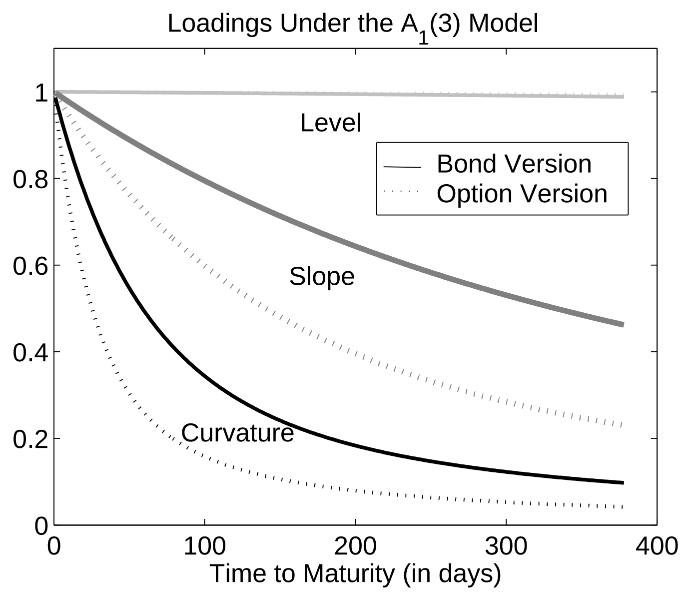

Figure 4: Loadings of the Three Dynamic Factors Under the $A_1(3)$ Model.

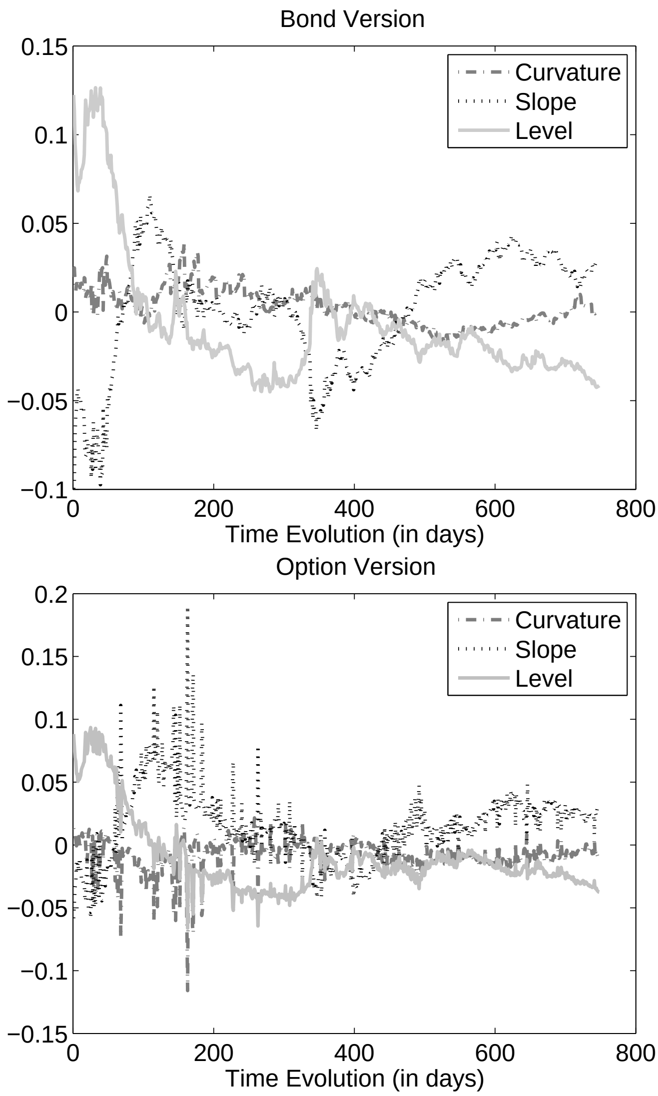

Figure 5: Time Series of the State Variables Under the Gaussian Model.

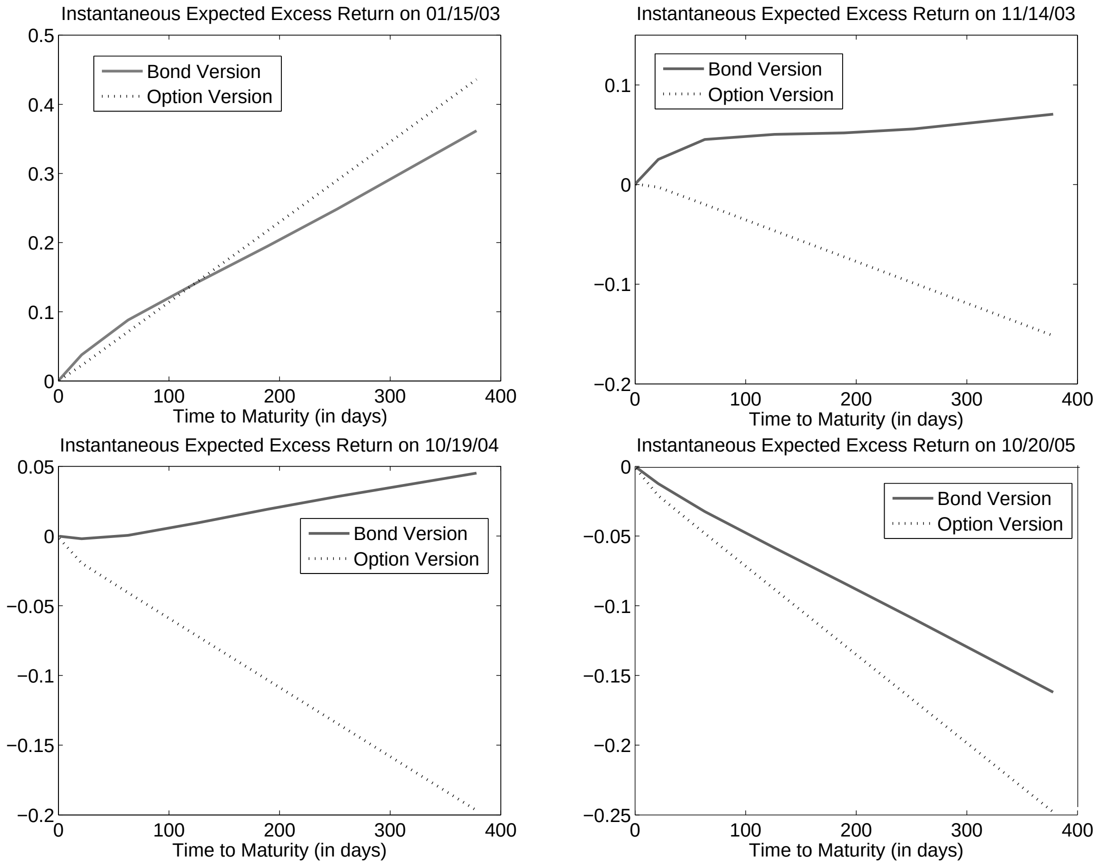

Figure 6: Examples of Cross-Section Instantaneous Expected Excess Returns Under the Gaussian Model.

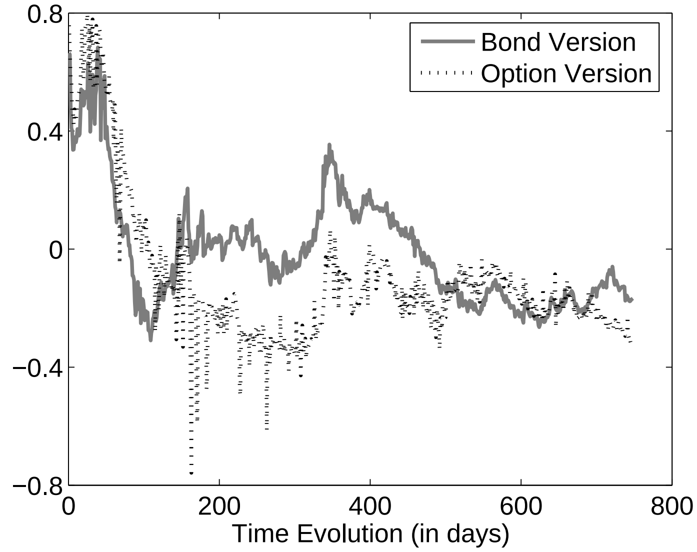

Figure 7: Time Series of Instantaneous Expected Excess Return for the 1-year Bond Under the Gaussian Model.

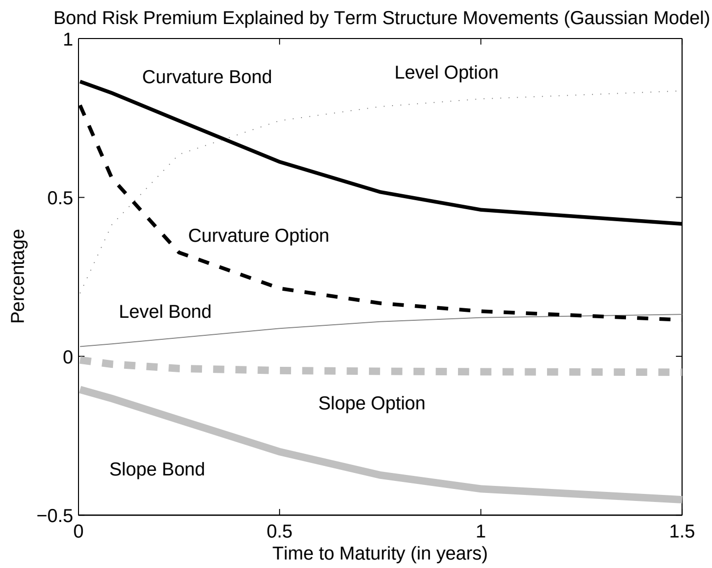

Figure 8: The Bond Risk Premium Decomposition for the Bond Version (Solid Line) and Option Version (Dashed Line) Under the Gaussian Model.

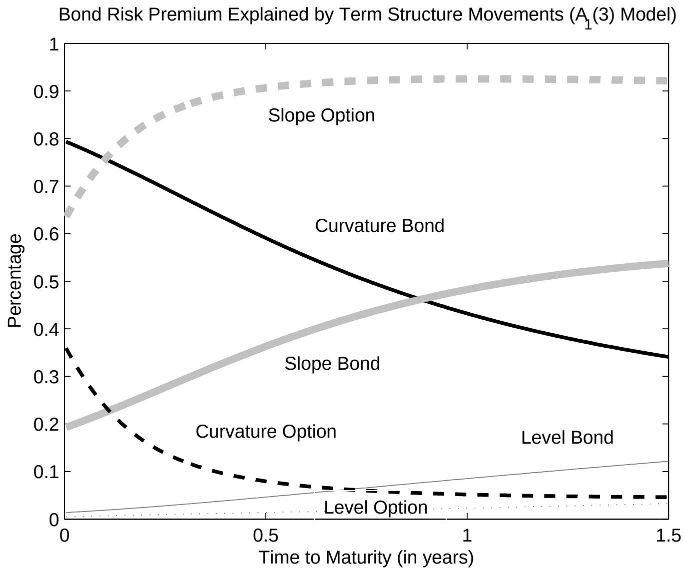

Figure 9: The Bond Risk Premium Decomposition for the Bond Version (Solid Line) and Option Version (Dashed Line) Under the $A_1(3)$ Model.

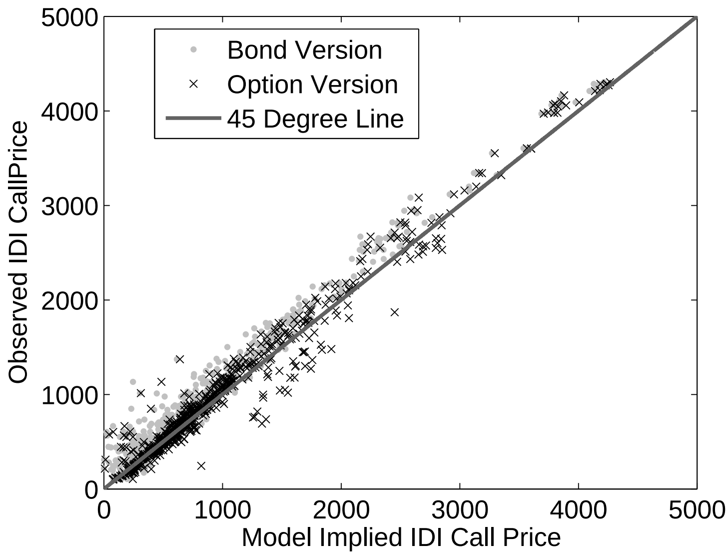

Figure 10: Observed IDI Call Price as a Linear Approximation of the Model-Implied Price Under the Gaussian Model

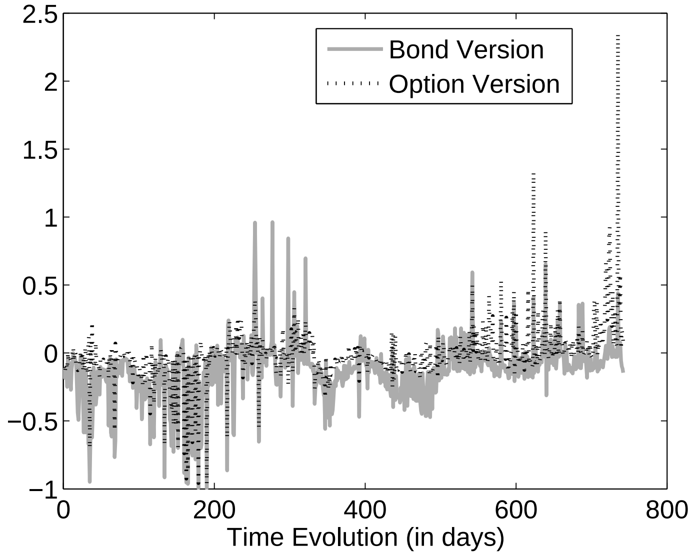

Figure 11: Model Relative Error when Pricing an IDI Call Based on Parameters Estimated Under the Bond Version (Solid Line) and Option Version (Dotted Line) (Gaussian Model).

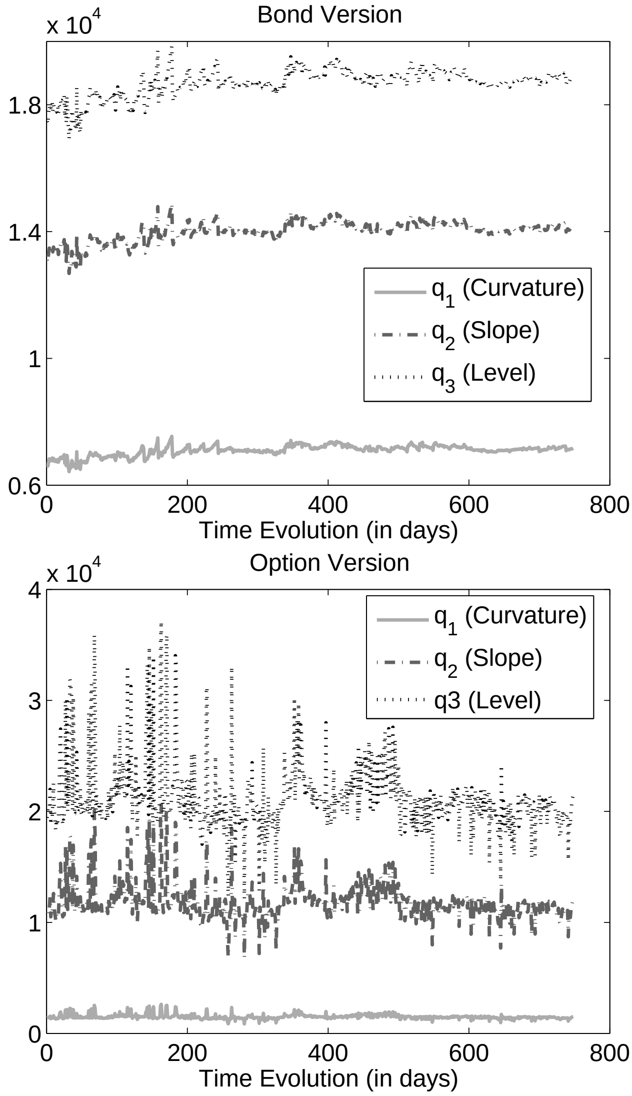

Figure 12: Units of State Variables in the Hedging Portfolio Under Both Versions of the Gaussian Model.