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Essays on Decision Theory

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Abstract

Even though completeness is a standard axiom for economic preferences, it is also a very strong assumption. According to von Neumann and Morgenstern (1953) (p. 631) “It is very dubious, whether the idealization of reality which treats this postulate as a valid one, is appropriate or even convenient.” in the same vein Aumann (1962) said “Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable”. This thesis aims to advance the study of incomplete preferences through three chapters.

Chapter 1 extends the notion of stochastic dominance to dynamic environments. Its main result is a clean characterization of the dominance orders between stochastic payoff processes given by unanimity within groups of discounted expected utility maximizers. A measure that quantifies the intensity of the dominance of a payoff process over another is defined and used to derive robust bounds on asset price differentials.

Chapter 2 extends Berge’s Maximum Theorem to allow for incomplete preferences. A simple version of the Maximum Theorem is provided for convex feasible sets and a fixed preference. Then, it shows that if, in addition to the traditional continuity assumptions, a new continuity property for the domains of comparability holds, the limits of maximal elements along a sequence of decision problems are maximal elements in the limit problem. While this new continuity property for the domains of comparability is sufficient, it is not generally necessary. However, conditions are given under which it is necessary and sufficient for maximality and minimality to be preserved by limits.

Chapter 3 defines and studies the class of “connected preferences”, that is, preferences that may fail to be complete but have connected maximal domains of comparability. It offers four new results. Theorem 3.1 identifies a basic necessary condition for a continuous preference to be connected in the sense above, while Theorem 3.2 provides sufficient conditions. Building on the latter, Theorem 3.3 characterizes the maximal domains of comparability. Finally, Theorem 3.4 presents conditions that ensure that maximal domains are arc-connected. Building on these results it is proven, for the case of compact spaces, a tight relationship between connectedness of the space and axioms on a preference defined on that space, in the spirit of the celebrated theorem of Schmeidler (1971).

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CHAPTER 1

Dynamic Stochastic Dominance

1.1. Introduction

First order stochastic dominance provides a robust decision rule for static economic models with uncertainty. If money lottery F first order stochastically dominates money lottery G, then any decision maker who maximizes expected utility and prefers more money to less would choose F over G. As a result, if several individuals with monotone preferences decide via consensus between lotteries F and G, we can expect them to choose F.

While the notion of first order stochastic dominance is essentially static, intertemporal trade-offs play a major role in economics. The main goal of this chapter is to bridge this gap by studying a class of orders between stochastic payoff processes which are naturally associated with robust decision rules for standard dynamic environments.

In most economic models, agents facing uncertainty make dynamic decisions by maximizing an objective function of the form $V(X|u,r) = \mathbb{E}\left\{r\int_0^{+\infty} e^{-rt}u(X_t)dt\right\}$, where X is a stochastic process which represents the payoff flow and $r > 0$ is a constant discount rate. Such agents will choose X over Y whenever $V(X|u,r)$ is larger than $V(Y|u,r)$. However, it is clear that specific hypothesis about u will affect choice behavior.

In order to identify comparisons which are robust to perturbations on vNM utility indexes, we define a class of a dynamic stochastic orders which essentially say that X dominates Y whenever $V(X|u,r)$ exceeds $V(Y|u,r)$ for every vNM index u satisfying certain conditions. Our main results reduce the analysis of dynamic stochastic dominance to static stochastic dominance of the relevant type, using an "average" probability distribution which combines the statistical properties of the payoffs processes and the intertemporal weighting (i.e., discounting) encoded in the agent's preferences. More specifically, Theorem 1.1 provides a clean characterization of the first order dynamic stochastic dominance order associated with nondecreasing vNM utility indexes, while Theorem 1.2 offers the same kind of characterization under the additional assumption of concavity.

Since the dynamic stochastic dominance orders we study naturally imply robust decision rules within groups of individuals, we can elicit how much we need to add to a payoff process X in order to dominate Y. The formalization of this idea leads to a family of "robust premia" that measure the intensity of dominance. These notions are useful because they provide a way to connect theoretical assumptions on preferences with observable data. Proposition 1.1 demonstrates this by using robust premia to obtain bounds on asset price differentials.

1.2. Related literature

The use of stochastic dominance as a decision rule to rank uncertain prospects in finance and economics originated in the 70's with Hadar and Russell (1971), Whitmore (1970) and Rothschild and Stiglitz (1970). Many developments have been made since then (see Levy (1992) and Levy and Kroll (1980) for an extensive review).

There exists some work that attempts to extend the notion of stochastic dominance to dynamic frameworks. Magnac and Robin (1999) define an alternative notion of dynamic stochastic dominance that also involves a general set of vNM utility indexes, but requires the existence of a transformation inducing static stochastic dominance for all t. Our definition seems more natural because it evaluates stochastic payoff flows using the most common model of preferences under uncertainty and does not require the existence of an abstract transformation. Arcand et al. (2020) propose a simple dynamic extension of the integral conditions that characterize static second order stochastic dominance. Their approach is limited, though, in that they only deal with second order stochastic dominance for Itô diffusions with constant volatility coefficient. In contrast, our approach is more general since it can accommodate dynamic stochastic dominance relations of any order and stochastic payoff processes of any kind.

This chapter is also related to the literature on model robustness regarding the specification of utility functions. Dubra et al. (2004a) axiomatization yields a preference that have a multi-utility representation, therefore in their framework a choice of one alternative over another is also represented by unanimity of that decision among a group of agents. Ok et al. (2012) axiomatize a preference that is robust to misspecification in distribution or in the utility function, whereas Galaabaatar and Karni (2012) allow for robustness in both dimensions at the same time. However, the main goal in this strand of literature is to axiomatize preferences and establish representation results. Instead, the present chapter starts from a discounted expected utility representation and uses it to define dynamic stochastic orders associated with robust decision rules.

1.3. Dynamic Stochastic Dominance

For a stochastic process $X \equiv \{X_t\}_{t\geq 0}$ we define

$$V(X|u,D) := \mathbb{E}\left\{\int_0^{+\infty} u(X_t)dD(t)\right\},$$

where $u : \mathbb{R} \to \mathbb{R}$ is a vNM utility index and $D : [0,+\infty) \to \mathbb{R}_+$ is a general intertemporal aggregator (i.e., the CDF associated to a probability measure weighting the different dates). We say that $V(X|u,D)$ is *well defined* when $V(X|u^+,D) < +\infty$ or $V(X|u^-,D) < +\infty$, where u^+ and u^- are the positive and negative parts of u, respectively.

Example 1.1. Let $D(t) := 1 - e^{-rt}$ for some fixed $0 < r < 1$. Then we have the standard expected discounted utility model with exponential discounting. That is,

$$V(X|u,D) := \mathbb{E}\left\{r\int_0^{+\infty} e^{-rt}u(X_t)dt\right\}.$$

Moreover, if u is bounded $V(X|u,D)$ is well defined.

Our first result shows that the expected discounted utility is equal to the expected utility taken with respect to a special CDF.

Lemma 1.1. *Let $F_X(\cdot|t)$ be the CDF of X_t. The discounted distribution of X is represented by the CDF*

$$G_X(x|D) := \int_0^{+\infty} F_X(x|t)dD(t).$$

Then

$$V(X|u,D) = \int_{-\infty}^{+\infty} u(x)dG_X(x|D)$$

for all measurable u such that $V(X|u,D)$ is well defined.

Proof. Let ν be the measure induced by $G_X(\cdot|D)$ and, for each t, μ_t be the measure induced by $F_X(\cdot|t)$. We start establishing a result for simple functions. Let $n \in \mathbb{N}$ and $\{A_i\}_{i=1}^n$ be a collection of measurable sets. Define $I(x) := \sum_{i=1}^n a_i \mathbb{1}_{A_i}(x)$, then

$$\begin{aligned}\mathbb{E}_\nu I = \sum_{i=1}^n a_i \nu(A_i) &= \sum_{i=1}^n a_i \int_0^{+\infty} \mu_t(A_i) dD(t) \\ &= \int_0^{+\infty} \sum_{i=1}^n a_i \mu_t(A_i) dD(t) \\ &= \int_0^{+\infty} (\mathbb{E}_{\mu_t} I)\, dD(t).\end{aligned}$$

Now let u be a measurable nonnegative function and I_n an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} I_n(x) = u(x)$ for every x. Thus, by definition and the previous result

$$\mathbb{E}_\nu u = \lim_{n\to\infty} \mathbb{E}_\nu I_n = \lim_{n\to\infty} \int_0^{+\infty} (\mathbb{E}_{\mu_t} I_n)\, dD(t).$$

Using the monotone convergence theorem we can pass the limit inside the integral to obtain $\mathbb{E}_\nu u = \int_0^{+\infty} (\mathbb{E}_{\mu_t} u)\, dD(t)$. Finally, for any measurable u such that $V(X|u^+, D) < +\infty$ or $V(X|u^-, D) < +\infty$

$$\begin{aligned}V(X|u, D) &= \mathbb{E}\left\{\int_0^{+\infty} u^+(X_t) dD(t)\right\} - \mathbb{E}\left\{\int_0^{+\infty} u^-(X_t) dD(t)\right\} \\ &= \int_0^{+\infty} \mathbb{E}u^+(X_t) dD(t) - \int_0^{+\infty} \mathbb{E}u^-(X_t) dD(t) \\ &= \mathbb{E}_\nu u^+ - \mathbb{E}_\nu u^- = \mathbb{E}_\nu u,\end{aligned}$$

where the second equality is an application of Tonelli's theorem, the third equality was proved in the previous step, and the last equality follows from the definition of the Lebesgue integral. □

Using this result we define an order on the space of stochastic processes.

DEFINITION 1.1. Let X and Y be two stochastic processes and $\mathcal{U}$ be a subset of functions $u : \mathbb{R} \to \mathbb{R}$. We say that X *$\mathcal{U}$-dynamically stochastically dominates* Y ($X \geq^{\mathcal{U}} Y$) whenever

$$V(X|u, D) \geq V(Y|u, D)$$

holds for every $u \in \mathcal{U}$ such that $V(X|u, D)$ and $V(X|u, D)$ are well defined.

Because $V(X|u, D)$ and $V(Y|u, D)$ need to be well defined, it is possible that only a proper subset of vNM utility indexes in $\mathcal{U}$ is used to check $V(X|u, D) \geq V(Y|u, D)$. More significantly, this subset may depend on the pair of stochastic processes that one wants to compare. This is an unfortunate feature of our definition, but one that is unrelated to dynamics (it is already present in the general versions of static first and second order stochastic dominance). Moreover, as we shall see, a single subset of $\mathcal{U}$ suffices to characterize the associated order in some contexts.

In the following subsections we discuss important instances of the general order, specified through restrictions on the set $\mathcal{U}$.

1.3.1. Dynamic first order stochastic dominance. Define

$$\mathcal{U}_1 := \{u : \mathbb{R} \to \mathbb{R} | u \text{ is nondecreasing}\}.$$

Whenever $X \geq^{\mathcal{U}_1} Y$, we say that X *first order dynamically stochastically dominates (FODSD)* Y . The next theorem is the main result of this subsection.

THEOREM 1.1. *The following are equivalent:*

(1) X first order dynamically stochastically dominates Y.

(2) $G_X(\cdot|D)$ first order stochastically dominates $G_Y(\cdot|D)$.

PROOF. Note that $G_X(\cdot|D)$ first order stochastically dominates $G_Y(\cdot|D)$ if and only if $\int_{-\infty}^{+\infty} u(x)dG_X(x|D) \geq \int_{-\infty}^{+\infty} u(x)dG_Y(x|D)$, for all nondecreasing u such that both integrals are well defined, then Lemma 1.1 yields the conclusion. □

REMARK 1.1. If process X is nonnegative we have $V(X|u^-, D) < +\infty$ for all $u \in \mathcal{U}_1$. Hence, for nonnegative stochastic processes the comparison between any pair of payoff processes relies on the same set of vNM utility indexes.

Theorem 1.1 establishes that first order dynamic stochastic dominance is equivalent to first order stochastic dominance between discounted CDFs. Theorem 1.1 also implies that having first order stochastic dominance between X_t and Y_t, for every t, is a sufficient condition for first order dynamic stochastic dominance. The next example illustrates how Theorem 1.1 can be used to check first order dynamic stochastic dominance while showing that having first order stochastic dominance between X_t and Y_t, for every t, is not a necessary condition for first order dynamic stochastic dominance.

EXAMPLE 1.2. Let X be a stochastic process such that $X_0 = 1$ and

$$X_t = \begin{cases} 1, & \text{with probability } e^{-t} \\ 0, & \text{with probability } 1 - e^{-t} \end{cases}$$

for $t > 0$. Let Y be a a stochastic process such that $Y_0 = 1$ and

$$Y_t = \begin{cases} 1, \text{ with probability } e^{-T} \\ 0, \text{ with probability } 1 - e^{-T} \end{cases}$$

for $t > 0$ and some fixed $T > 0$. Let $D(t) = 1 - e^{-rt}$ for some fixed $0 < r < 1$. Note that

$$F_X(x|t) = \begin{cases} 1 & , \text{ if } x \geq 1 \\ 1 - e^{-t} & , \text{ if } 0 \leq x < 1 \\ 0 & , \text{ if } x < 0 \end{cases}$$

and

$$F_Y(x|t) = \begin{cases} 1 & , \text{ if } x \geq 1 \\ 1 - e^{-T} & , \text{ if } 0 \leq x < 1 \\ 0 & , \text{ if } x < 0. \end{cases}$$

Then $F_X(\cdot|t)$ dominates $F_Y(\cdot|t)$ in first order stochastic sense for every $t \leq T$ and vice versa for $t > T$. Evaluating the discounted distribution of each process we obtain

$$G_X(x|D) = \begin{cases} 1 & , \text{ if } x \geq 1 \\ 1 - \frac{r}{r+1} & , \text{ if } 0 \leq x < 1 \\ 0 & , \text{ if } x < 0 \end{cases}$$

and

$$G_Y(x|D) = \begin{cases} 1 & , \text{ if } x \geq 1 \\ 1 - e^{-T} & , \text{ if } 0 \leq x < 1 \\ 0 & , \text{ if } x < 0. \end{cases}$$

Define $\bar{G}_X(r) := 1 - \frac{r}{r+1}$ and $\bar{G}_Y(T) := 1 - e^{-T}$. Figure 1.1 shows in red the region where the combination of parameters r and T makes X FODSD Y and in blue the region where Y FODSD X.

Hence, we can see that there are parameters T and r such that the worse initial distributions of payoffs Y are compensated by the worsen in distribution that happens with payoff X, in a way that every monotonic decision maker would prefer Y over X.

1.3.2. Dynamic second order stochastic dominance. Define

$$\mathcal{U}_2 := \{u : \mathbb{R} \to \mathbb{R} | u \text{ is nondecreasing and concave}\}.$$

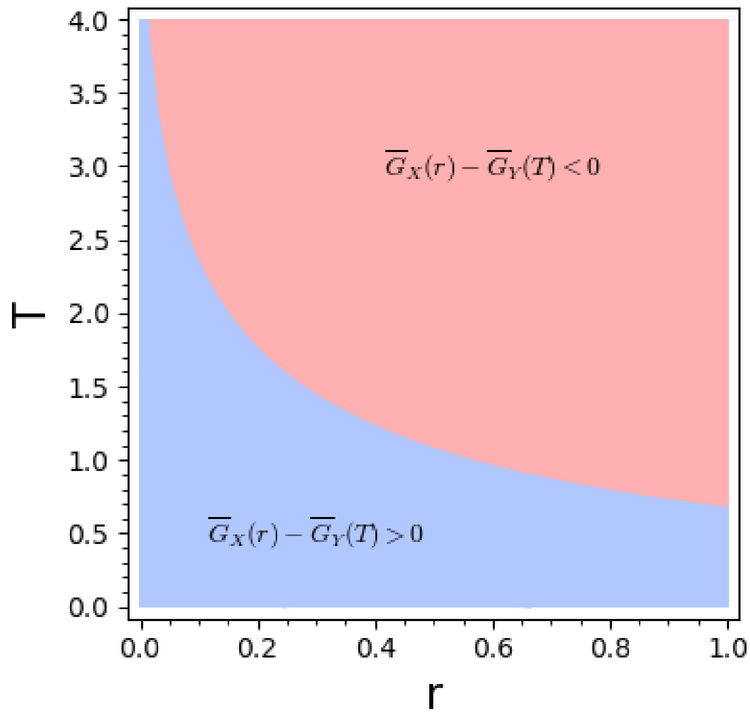

FIGURE 1.1. Example 1.2: FODSD between X and Y

Whenever $X \geq^{\mathcal{U}_2} Y$, we say that X *second order dynamically stochastically dominates (SODSD)* Y. We can establish a result that parallels Theorem 1.1.

THEOREM 1.2. *The following are equivalent:*

(1) X *second order dynamically stochastically dominates* Y*.*
(2) $G_X(\cdot|D)$ *second order stochastically dominates* $G_Y(\cdot|D)$*.*

PROOF. We know that $G_X(\cdot|D)$ second order stochastically dominates $G_Y(\cdot|D)$ if and only if $\int_{-\infty}^{+\infty} u(x)dG_X(x|D) \geq \int_{-\infty}^{+\infty} u(x)dG_Y(x|D)$, for all nondecreasing and concave u such that both integrals are well defined, then we only need to apply Lemma 1.1. □

1.4. Robust premium

In this section, we use first order dynamic stochastic dominance to construct a measure that quantifies relative dominance between stochastic payoff processes.

DEFINITION 1.2. Let X and Y be two stochastic processes. We define the *robust premium* as

$$RP(X,Y) := \inf\{\epsilon > 0 |\, (Y+\epsilon) \text{ FODSD } X\}.$$

Intuitively, the robust premium is the minimum annuity that one needs to add to the payoff flow Y, such that the resulted shifted process FODSD X. Because of Theorem 1.1, the robust premium is the minimum shift to the right on G_Y that makes it first order stochastically dominates G_X.

REMARK 1.2. Let Y and Z be two stochastic payoff processes. If Y FODSD Z, then $RP(Y, X) \geq RP(Z, X)$, for any process X. This is so because Theorem 1.1 implies that first order dynamic stochastic dominance is a transitive order. Moreover, $X + RP(Y, X)$ FODSD Y by the definition of robust premium. These two facts together imply that $X + RP(Y, X)$ FODSD Z.

We could have defined the robust premium in terms of the general $\mathcal{U}$-dynamic dominance, but much of its interpretation would be lost, since without restricting ourselves to nondecreasing u adding a positive annuity to a payoff flow could harm the decision maker.

Another possibility would be to use second order dynamic stochastic dominance instead of first order dynamic stochastic dominance. Since its interpretation and the relation with second order dynamic stochastic dominance are completely analogous to the case of first order dynamic stochastic dominance, we omit the details.

The next example shows how Theorem 1.1 makes simple the assessment of the robust premium.

EXAMPLE 1.3. Keep X, Y and D as defined in Example 1.2. We can choose r and T such that $\bar{G}_Y(T) > \bar{G}_X(r)$. We plot these two CDFs together in Figure 1.2.

Since the discounted CDF of $Y + \epsilon$ is just the discounted CDF of Y shifted to the right by ϵ, $RP(X, Y)$ will be the minimum shift to the right that makes the shifted CDF stays below G_X for every point. Hence, $RP(X, Y) = 1$.

On the other hand, G_X already dominates G_Y in first order stochastic sense, then no shift to the right is necessary implying $RP(Y, X) = 0$.

Another interesting aspect of the robust premium is that it allows for comparison between stochastic processes that could not be ordered in first order dynamic stochastic sense before.

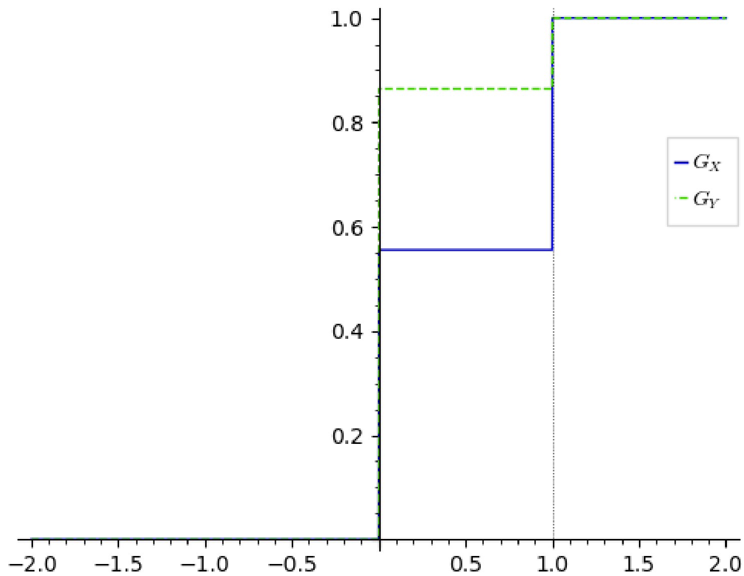

FIGURE 1.2. Example 1.3: representation of G_X and G_Y

EXAMPLE 1.4. Let's keep process X and D as in Example 1.2, but change Y to be $Y_0 = 1$ and

$$Y_t = \begin{cases} 1.3, & \text{with probability } e^{-T} \\ 0.3, & \text{with probability } 1 - e^{-T} \end{cases}$$

for $t > 0$ and some fixed $T > 0$. Of course G_X is the same as before, but

$$G_Y(x|D) = \begin{cases} 1 & , \text{ if } x \geq 1.3 \\ 1 - e^{-T} & , \text{ if } 0.3 \leq x < 1.3 \\ 0 & , \text{ if } x < 0.3. \end{cases}$$

Note that this new G_Y is the former one shifted to the right by 0.3. Figure 1.3 shows the graph of G_X and G_Y when we choose r and T such that $\bar{G}_Y > \bar{G}_X$. Using again Theorem 1.1 it is easy to see that $RP(X,Y) = 0.7$ and $RP(Y,X) = 0.3$. Note that $RP(X,Y)$ is less than in Example 1.3, because G_Y here dominates in first order stochastic sense the G_Y from Example 1.3. By the same reasoning $RP(Y,X)$ is larger now.

The next result shows that the robust premium can be used to bound the difference between asset prices.

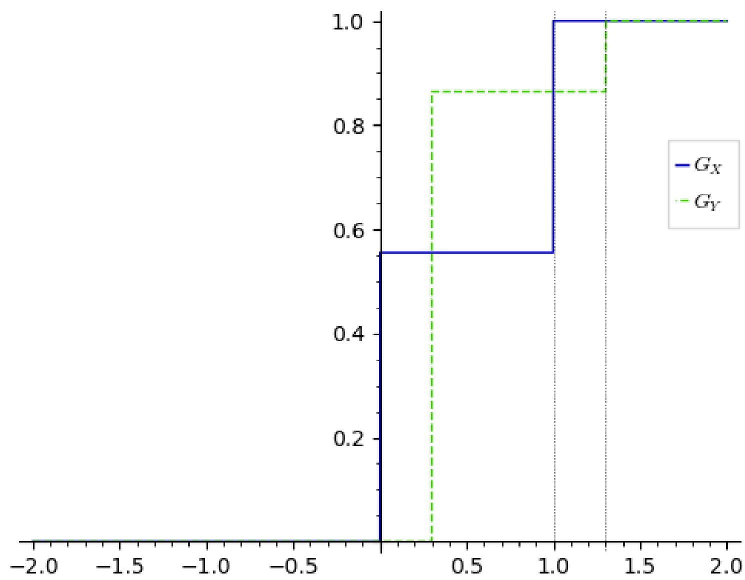

FIGURE 1.3. Example 1.4: representation of G_X and G_Y

PROPOSITION 1.1. *Consider a Lucas economy in continuous time, with complete markets, risk free rate $r > 0$, unlimited short sales, infinitely many agents, and positive endowments for everyone. Each individual i preference can be represented by*

$$V_i(S) := \mathbb{E}\left\{r\int_0^{+\infty} e^{-rt}u_i\left(S_t^i\right)dt\right\},$$

where S_t^i is his consumption for each t and u_i is nondecreasing for all i. Let $X = \{X_t\}_{t>0}$ and $Y = \{Y_t\}_{t>0}$ be two assets with prices $P^X = \left\{P_t^X\right\}_{t>0}$ and $P^Y = \left\{P_t^Y\right\}_{t>0}$ respectively. Then,

$$P_0^Y - \frac{RP(Y,X)}{r} \leq P_0^X \leq P_0^Y + \frac{RP(X,Y)}{r}.$$

PROOF. Note that if $RP(Y,X) = +\infty$ or $RP(X,Y) = +\infty$ the corresponding inequalities hold trivially. Thus, assume that both premia are finite. Given our setup, arbitrage cannot exist in equilibrium. First we will prove that $P_0^X \leq P_0^Y + \frac{RP(X,Y)}{r}$. Suppose, seeking a contradiction, that $P_0^X - P_0^Y > \frac{RP(X,Y)}{r}$, then agent i^*, at $t = 0$, could sell short one unit of asset X and buy one unit of asset Y. The same agent can propose to any other agent, say i, to pay $Y_t + RP(S,Y)$ to i, for all $t > 0$, in exchange for receiving S_t each period from i. Since $V_i(Y + RP(S,Y)) \geq V_i(S)$, for all i, any agent in this economy would accept this deal. In the end, the net position of agent i^* is an annuity of $RP(S,Y)$ that he has to pay, which has present value of $\frac{RP(S,Y)}{r}$. Hence, in $t = 0$, he gains $P_0^S - P_0^Y - \frac{RP(S,Y)}{r} > 0$, a contradiction with no arbitrage.

To prove $P_0^Y - \frac{RP(Y,S)}{r} \leq P_0^S$, one could reverse the strategy already described, also yielding a contradiction. □

It is important to note that the robust premium can be $+\infty$, because first order dynamic stochastic dominance is equivalent to the static first order stochastic dominance and there are distributions that are incomparable in first order stochastic dominance sense, no matter how much we shift to the right one of the CDFs. Normal distributions with different volatilities are a good example. In this case, Proposition 1.1 imposes no bounds on the difference of asset prices.

If we have used second order dynamic stochastic dominance in the definition of Robust Premium there would be less cases of it being $+\infty$, since first order stochastic dominance implies second order stochastic dominance, but the converse does not hold. Moreover, because of this relation between stochastic dominance orders, a Robust Premium based on second order dynamic stochastic dominance would be lower than a Robust Premium based on first order stochastic dominance. Hence, using second order dynamic stochastic dominance to define the Robust Premium would imply tighter bounds on price differentials.

1.5. Discussion

In this chapter we define general orders on the space of stochastic processes that naturally represent a decision rule among payoff processes and we offer a full characterization to two of these orders, namely first (and second) order dynamic stochastic dominance, which are natural extensions of first (and second) order stochastic dominance. We also define a measure of the relative dominance between stochastic payoff processes and Proposition 1.1 shows that this measure disciplines the relation of theoretical preference assumptions and observable data on asset prices.

As we have established, our dynamic stochastic orders imply decision rules that are robust to specific assumptions about vNM utility indexes. One extension is to define orders that not only relate to utility function robustness, but also relate to discounting function robustness. More concretely,

Definition 1.1′. Let X and Y be two stochastic processes. Let $\mathcal{U}$ and $\mathcal{B}$ be, respectively, a subset of functions $u : \mathbb{R} \to \mathbb{R}$ and a subset of CDFs $D : [0, +\infty) \to \mathbb{R}_+$. We say that X *$\mathcal{U}$-dynamically stochastically dominates* Y ($X \geq^{\mathcal{U}} Y$) whenever

$$V(X|u, D) \geq V(Y|u, D)$$

holds for every $u \in \mathcal{U}$ and every $D \in \mathcal{B}$ such that $V(X|u, D)$ and $V(Y|u, D)$ are well defined.

Then the content of Theorem 1.1 could be rewritten as

THEOREM 1.1′. *The following are equivalent:*

(1) X first order dynamically stochastically dominates Y.
(2) $G_X(\cdot|D)$ first order stochastically dominates $G_Y(\cdot|D)$, for all $D \in \mathcal{B}$.

With the same sort of adaptation to Theorem 1.2. A natural route for further research is to investigate what are the implications of first and second orders dynamic stochastic dominance to optimal stopping problems and control problems in a dynamic framework. Related to that, still is an open question if our dynamic stochastic orders allow comparative statics of optimal stopping times in the spirit of Quah and Strulovici (2013).

CHAPTER 2

Maximum Theorem for Incomplete Preferences

2.1. Introduction

An important issue arising in the study of models involving optimization is whether optimal choices depend continuously on parameters affecting the objective function and the constraints. The main tool to address this question is the Maximum Theorem by Berge (1963), which can be stated as follows:

Maximum Theorem. *Let X and Θ be topological spaces; let $u : X \times \Theta \to \mathbb{R}$ be a continuous function; let $K : \Theta \rightrightarrows X$ be a continuous and compact-valued correspondence. Then, the correspondence $M : \Theta \rightrightarrows X$ defined by setting $M(\theta) := \arg\max_{x \in K(\theta)} u(x, \theta)$ for each $\theta \in \Theta$ is upper hemicontinuous and compact-valued.*

This result can be easily modified to dispense with utility functions and deal directly with complete (continuous) preferences and also with incomplete preferences with open asymmetric parts (see Walker (1979)). However, to the best of our knowledge, none of the existing generalizations of the Maximum Theorem applies to some of the most standard types of incomplete preferences such as Pareto orderings based on continuous utility functions, preferences over lotteries admitting an expected multi-utility representation as in Dubra, Maccheroni, and Ok (2004b), or ordinal preferences possessing a continuous multi-utility representation studied by Evren and Ok (2011).

The following example shows that obtaining a maximum theorem that covers these types of preferences requires additional conditions:

Example 2.1. Consider a consumer choosing bundles of two goods: apples (A) and bananas (B). Her preferences are fixed, but incomplete. They can be represented with two utilities: $u_1(q_A, q_B) = q_A + q_B$ and $u_2(q_A, q_B) = q_A + 2q_B$, in the sense that a bundle (q_A, q_B) is considered at least as good as another bundle (q'_A, q'_B) if and only if $u_1(q_A, q_B) \geq u_1(q'_A, q'_B)$ and $u_2(q_A, q_B) \geq u_2(q'_A, q'_B)$. The price of apples, p_A, is normalized to 1 and there is sequence of prices for bananas $p_{B,n} = 1 + 1/n$. Note that, if the consumer's wealth is $w = 1$, bundle $(1, 0)$ is optimal for every $n \in \mathbb{N}$: there is no feasible bundle that the consumer strictly prefers to $(1, 0)$. However, in the limit

$n \to +\infty$, the bundle $(1,0)$ is no longer optimal because the consumer strictly prefers the bundle $(0,1)$, which is feasible when the prices are $(p_A, p_{B,+\infty}) = (1,1)$.

The example above is fairly simple and suggests that, when weak preferences are continuous but incomplete, we should not be surprised to find sequences of maximal elements that converge to suboptimal alternatives. With this observation in mind, the main contributions of the present chapter are: 1) to provide conditions which ensure that limits of optimal choices are optimal in the limit problem, and 2) to shed light on how the nature of preference incompleteness may interfere with the preservation of optimality when taking limits.

The rest of the paper is organized as follows. We present basic definitions in Section 2.2. We state and prove a simple Maximum Theorem for incomplete preferences in Section 2.3. This result is somewhat restrictive since it requires a fixed preference and convex feasible sets. In Section 2.4, we establish a more general Maximum Theorem based on a new continuity condition for the domains of comparability of the preferences involved. In Section 2.5, we investigate assumptions under which this new condition is not only sufficient but also essentially necessary. Finally, Section 2.6 briefly relates our results to the existing literature and Section 2.7 offers some concluding remarks. We relegated all auxiliary lemmas and their proofs to Appendix A.

2.2. Preliminaries

Let (X,d) be a metric space. In this thesis, a *preference*, generically denoted by $\succsim$, is a reflexive and transitive binary relation on X. As usual, $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$, respectively. For every $x \in A \subseteq X$, the set $\{y \in A | y \sim x\}$ is the *indifference class* of $\succsim$ in A.

We say that $\succsim$ is *complete* on a set $A \subseteq X$ if either $x \succsim y$ or $y \succsim x$ holds for all $x, y \in A$. The set A is a $\succsim$*-domain* if $\succsim$ is complete on A. If $A \subseteq B \subseteq X$ and A is a $\succsim$-domain such that there exists no $\succsim$-domain contained in B and strictly containing A, then A is a *maximal* $\succsim$*-domain relative to* B. Denote by $\mathcal{D}(\succsim, B)$ the collection of all maximal $\succsim$-domains relative to B.

A point $x \in A$ is $\succsim$*-maximal in* A if, for every $y \in A$, $y \succsim x$ implies $x \succsim y$. The set of all $\succsim$-maximal elements in A is denoted by $\mathcal{M}ax(\succsim, A)$. Analogously, a point $x \in A$ is $\succsim$*-minimal in* A if, for every $y \in A$, $x \succsim y$ implies $y \succsim x$. The set of all $\succsim$-minimal elements in A is denoted by $\mathcal{M}in(\succsim, A)$.

A preference $\succsim$ is *continuous* if it is a closed subset of $X \times X$. Let $\mathcal{P}$ be the collection of continuous preferences on X. A set $\mathcal{U} \subseteq \mathbb{R}^X$ is a *multi-utility representation* for $\succsim$ whenever, for every $x, y \in X$, $x \succsim y$ holds if and only if $u(x) \geq u(y)$ for all $u \in \mathcal{U}$.

Let $\mathcal{K}_X$ be the collection of nonempty compact subsets of X. Consider both $\mathcal{K}_X$ and $\mathcal{P}$ equipped with the Hausdorff metric topology derived from X and $X \times X$, respectively. Finally, for any sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of nonempty subsets of $\mathcal{K}_X$, denote by $LS_{n\to+\infty}\mathcal{A}_n$ the collection of accumulation points of all sequences $\{A_n\}_{n\in\mathbb{N}}$, where $A_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$.

2.3. A Simple Maximum Theorem

In this section, we introduce a new continuity condition that is compatible with interesting classes of incomplete preferences and allows us to prove a simple Maximum Theorem. Throughout this section, we will assume that X is convex.

DEFINITION 2.1. A preference $\succsim$ is *midpoint continuous* if, for every $x, y \in X$ satisfying $y \succ x$, there exists $\alpha \in [0, 1)$ and open sets $V, W \subseteq X$ such that $(\alpha x + (1-\alpha)y, x) \in V \times W$ and $z' \succ x'$ for all $(z', x') \in V \times W$.

Every complete and continuous preference satisfies midpoint continuity.[1] The following result establishes that a significant class of incomplete preferences also does:

PROPOSITION 2.1. *If $\succsim$ admits a finite multi-utility representation $\mathcal{U} \subseteq \mathbb{R}^X$ with each $u \in \mathcal{U}$ continuous and strictly quasiconcave, then $\succsim$ satisfies midpoint continuity.*

PROOF. Take $x, y \in X$ such that $x \succ y$. By definition of multi-utility representation, we have $u(x) \geq u(y)$ for all $u \in \mathcal{U}$. Define $z := (1/2)x + (1/2)y$. Since each $u \in \mathcal{U}$ is strictly quasi-concave, $u(z) > u(y)$ holds for all $u \in \mathcal{U}$. For each $u \in \mathcal{U}$, there are open sets V_u, W_u such that $(z, y) \in V_u \times W_u$ and $u(z') > u(y')$ for all $(z', y') \in V_u \times W_u$. Define $V := \bigcap_{u\in\mathcal{U}} V_u$ and $W := \bigcap_{u\in\mathcal{U}} W_u$. Clearly $(z, y) \in V \times W$, so V and W are nonempty. Moreover, since $\mathcal{U}$ is finite, V and W are open. By construction, we have $u(z') > u(y')$ for all $z' \in V$, $y' \in W$, and $u \in \mathcal{U}$. Since $\mathcal{U}$ is a multi-utility representation, $(z', y') \in V \times W$ implies $z' \succ y'$, showing that $\succsim$ satisfies midpoint continuity. □

Using the concept of midpoint continuity, we can establish the first major result of this chapter, a simple Maximum Theorem:

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[1] If a preference is complete and continuous, its asymmetric part is open in $X \times X$. As a result, we can always take $\alpha = 0$ to satisfy the de nition of midpoint continuity.

THEOREM 2.1. *Let* $\succsim \in \mathcal{P}$ *and let* $\{(K_n, x_n)\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{K}_X \times X$ *such that*

(1) $\{(K_n, x_n)\}_{n\in\mathbb{N}}$ *converges to* $(K, x) \in \mathcal{K}_X \times X$ *as* $n \to +\infty$*.*
(2) $x_n \in \mathcal{M}ax\,(\succsim, K_n)$ *for every* $n \in \mathbb{N}$*.*
(3) K_n *is convex for every* $n \in \mathbb{N}$*.*
(4) $\succsim$ *satisfies midpoint continuity.*

Then, $x \in \mathcal{M}ax\,(\succsim, K)$*.*

PROOF. Suppose, seeking a contradiction, that $x \notin \mathcal{M}ax\,(\succsim, K)$. Then, there exists $y \in K$ such that $y \succ x$. By midpoint continuity, there must exists $\alpha \in [0, 1)$ and open sets $V, W \subseteq X$ satisfying $(\alpha x + (1-\alpha)y, x) \in V \times W$ and $z' \succ x'$ for all $(z', x') \in V \times W$. Define $z := \alpha x + (1-\alpha)y$. Note that, being the limit of a sequence of convex sets, K must be convex. It follows that $z \in K$. Since $x = \lim_{n\to+\infty} x_n$, there must be $N_1 \in \mathbb{N}$ such that $x_n \in W$ for all $n \geq N_1$. Since $K = \lim_{n\to+\infty} K_n$ and $z \in K$, there must be $N_2 \in \mathbb{N}$ such that $K_n \cap V \neq \emptyset$ for all $n \geq N_2$. Take $N := \max\{N_1, N_2\}$. Then, for all $n \geq N$, there exists $z_n \in K_n$ such that $z_n \succ x_n$. This implies $x_n \notin \mathcal{M}ax\,(\succsim, K_n)$, a contradiction. We conclude that $x \in \mathcal{M}ax\,(\succsim, K)$. □

Note that the preferences in Example 2.1 violate midpoint continuity. In the following example, we apply Proposition 2.1 and Theorem 2.1 to establish the upper hemicontinuity of Pareto efficient allocations with respect to resource endowments.

EXAMPLE 2.2. Consider a simple exchange economy with $L \in \mathbb{N}$ goods and $N \in \mathbb{N}$ agents. Here $X = \mathbb{R}_+^{L\times N}$ denotes the set of all possible allocations or social outcomes. Each agent has a continuous and strictly convex preference on $\mathbb{R}_+^L$.[2] By definition, the Pareto relation for this economy admits a finite multi-utility representation composed by continuous and strictly quasiconcave utilities. It thus follows from Proposition 2.1 that the Pareto relation on X satisfies midpoint continuity. Now suppose that we have a sequence of endowments $\{\omega_n\}_{n\in\mathbb{N}}$ that converges to ω and a sequence of allocations $\{x_n\}_{n\in\mathbb{N}}$ that converges to an allocation x. By Theorem 2.1, if each x_n is Pareto efficient when the endowment is ω_n, then x is Pareto efficient when the endowment is ω.

Theorem 2.1 deals with a fixed preference $\succsim$. It would be desirable to relax this requirement, allowing for a sequence of preferences converging to $\succsim$. However, the following example shows that Theorem 2.1 becomes false with this modification.

[2] A preference is said to be *strictly convex* if, for every $x, y, z \in X$, $y \succsim x$, $z \succsim x$, $y \neq z$, and $\alpha \in (0, 1)$ imply $\alpha y + (1-\alpha)z \succ x$. See Mas-Colell, Whinston, and Green (1995), p. 44.

EXAMPLE 2.3. Let $X = [0,1]$ and let $\succsim_n$ be represented by $\mathcal{U}_n := \{u, v_n\}$, where $u(x) = x$ and $v_n(x) = \left(x - \frac{n+1}{2n}\right)^2$. Clearly, $\succsim_n$ converges to $\succsim$, the preference represented by $\mathcal{U} := \{u, v\}$, where $v(x) = \left(x - \frac{1}{2}\right)^2$. Moreover, every preference considered is continuous and satisfies midpoint continuity because of Proposition 2.1. However, even though $0 \in \mathcal{M}ax\,(\succsim_n, X)$ for each $n \in \mathbb{N}$, $0 \notin \mathcal{M}ax\,(\succsim, X)$.

2.4. A General Maximum Theorem

A significant limitation of Theorem 2.1 is that it requires a fixed preference and convex feasible sets. The second major result of this chapter replaces these restrictions with a continuity condition on maximal domains of comparability:

THEOREM 2.2. *Let $\{(\succsim_n, K_n, x_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P} \times \mathcal{K}_X \times X$ such that*

(1) $\{(\succsim_n, K_n, x_n)\}_{n\in\mathbb{N}}$ converges to $(\succsim, K, x) \in \mathcal{P} \times \mathcal{K}_X \times X$ as $n \to +\infty$.
(2) $x_n \in \mathcal{M}ax\,(\succsim_n, K_n)$ for every $n \in \mathbb{N}$.
(3) $LS_{n\to+\infty}\mathcal{D}\,(\succsim_n, K_n) \subseteq \mathcal{D}\,(\succsim, K)$.

Then, $x \in \mathcal{M}ax\,(\succsim, K)$.

PROOF. For each $n \in \mathbb{N}$, there exists $D_n \in \mathcal{D}\,(\succsim_n, K_n)$ such that $x_n \in D_n$. Since $K_n \in \mathcal{K}_X$ converges to $K \in \mathcal{K}_X$ and D_n is closed in K_n, we have $D_n \in \mathcal{K}_X$. By Lemma A.1, there exists a convergent subsequence $(D_{n_h})_{h\in\mathbb{N}}$. As a result, there is no loss of generality in assuming that $\{D_n\}_{n\in\mathbb{N}}$ itself converges. Define $D := \lim_{n\to+\infty} D_n$. By Lemma A.3, $x_n \in D_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty}(D_n, x_n) = (D, x)$ together imply that $x \in D$. Moreover, condition (3) implies that $D \in \mathcal{D}\,(\succsim, K)$.

We now claim that x is a $\succsim$-best in D. To prove this, suppose, seeking a contradiction, that there exists $y \in D$ such that $y \succ x$. Since $\lim_{n\to+\infty} D_n = D$, there must exist a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} y_n = y$ and $y_n \in D_n$ for every $n \in \mathbb{N}$. Moreover, since $\lim_{n\to+\infty} \succsim_n = \succsim$, by the second part of Lemma A.3, there must exist $N \in \mathbb{N}$ such that $y_N \succ_N x_N$. This contradicts that x_N is $\succsim_N$-maximal in K_N, as assumed.

Since x is $\succsim$-best in $D \in \mathcal{D}\,(\succsim, K)$, Theorem 1 of Gorno (2018) implies that x is $\succsim$-maximal in K. □

Theorem 2.2 generalizes the upper-hemicontinuity of the arg max correspondence in Berge's Maximum Theorem by weakening the completeness implied by the existence of a utility representation to condition (3). Roughly, this condition says that limits of maximal $\succsim_n$-domains should be maximal $\succsim$-domains, relative to the relevant feasible sets. In the particular case in which all preferences in the sequence $\{\succsim_n\}_{n\in\mathbb{N}}$

are complete, the limit preference $\succsim$ must also be complete and condition (3) holds trivially. However, condition (3) is also compatible with incomplete preferences.

Example 2.4. Suppose there is a finite partition $\mathcal{D}^*$ of X such that $\mathcal{D}(\succsim_n, X) = \mathcal{D}^*$ for all $n \in \mathbb{N}$. Note that this assumption nests the case of complete preferences as the particular case in which $\mathcal{D}^* = \{X\}$. Convergence of preferences implies that $\mathcal{D}(\succsim, X) = \mathcal{D}^*$ as well. Moreover, since all maximal domains relative to X are disjoint, we also have $\mathcal{D}(\succsim, K) = \{D \cap K | D \in \mathcal{D}^*\}$ and $\mathcal{D}(\succsim_n, K_n) = \{D \cap K_n | D \in \mathcal{D}^*\}$ for every $n \in \mathbb{N}$. We conclude that $LS_{n \to +\infty}\mathcal{D}(\succsim_n, K_n) \subseteq \mathcal{D}(\succsim, K)$ and condition (3) holds.

2.5. A Characterization

Even though condition (3) in Theorem 2.2 constitutes a general sufficient condition for optimality to be preserved by limits, it is not necessary:

Example 2.5. Let $X = [0, 1]$. Consider the following preference

$$\succsim = \left\{(x, y) \in [0, 0.5)^2 \middle| x = y\right\} \cup [0.5, 1]^2.$$

Consider the sequence $\{K_n\}_{n\in\mathbb{N}}$, where $K_n := [0.5 - 0.5/n, 1]$ for each $n \in \mathbb{N}$. On the one hand, $D_n = \{0.5 - 0.5/n\}$ is a maximal $\succsim$-domain relative to K_n, while the sequence $\{D_n\}_{n\in\mathbb{N}}$ converges to $D = \{0.5\}$, which is not a maximal $\succsim$-domain relative to $K := \lim_{n\to+\infty} K_n = [0.5, 1]$. On the other hand, $\mathcal{M}ax(\succsim_n, K_n) = \mathcal{M}in(\succsim_n, K_n) = K_n$ for all $n \in \mathbb{N}$ and $\mathcal{M}ax(\succsim, K) = \mathcal{M}in(\succsim, K) = K$, so all convergent sequences composed by $\succsim$-maximal and $\succsim$-minimal elements in each K_n converge to $\succsim$-maximal and $\succsim$-minimal elements in K.

However, condition (3) is indeed necessary and sufficient for maximal and minimal elements to be preserved by limits in more specific settings. In this section, we obtain a characterization by restricting attention to limit preferences that are antisymmetric (*i.e.*, partial orders) and sets that are "order dense".

Formally, a set $A \subseteq X$ is $\succsim$-*dense* if, for every $x, y \in A$, $x \succ y$ implies that there exists $z \in A$ such that $x \succ z \succ y$. $\succsim$-dense sets are quite common in applications. For instance, if $\succsim$ is a preference over lotteries that admits an expected multi-utility representation[3], then every convex set of lotteries is $\succsim$-dense. We can now state the main result of this section.

[3]Dubra, Maccheroni, and Ok (2004b) show that a preference over lotteries has an expected multi-utility representation if and only if it is continuous and satis es the independence axiom.

THEOREM 2.3. *Denote by $\mathcal{G} \subseteq \mathcal{P}$ the collection of continuous partial orders on X. Let $\{(\succsim_n, K_n)\}_{n\in\mathbb{N}}$ be a converging sequence in $\mathcal{P} \times \mathcal{K}_X$ with limit $(\succsim, K) \in \mathcal{G} \times \mathcal{K}_X$ and such that, for every $n \in \mathbb{N}$, K_n is a $\succsim_n$-dense set and all indifference classes of $\succsim_n$ in K_n are connected. Then, K is $\succsim$-dense. Moreover, the following are equivalent:*

(1) $LS_{n\to+\infty}\mathcal{D}(\succsim_n, K_n) \subseteq \mathcal{D}(\succsim, K)$
(2) $LS_{n\to+\infty}\mathcal{M}ax(\succsim_n, K_n) \subseteq \mathcal{M}ax(\succsim, K)$ and $LS_{n\to+\infty}\mathcal{M}in(\succsim_n, K_n) \subseteq \mathcal{M}in(\succsim, K)$.

PROOF. (1) $\Rightarrow$ (2). The convergence of the maximal elements is a direct implication of Theorem 2.2. To see the convergence of the minimal elements let $\{y_n\}_{n\in\mathbb{N}}$ be a convergent sequence such that $\lim_{n\to+\infty} y_n = y$ and, for all $n \in \mathbb{N}$, y_n is a $\succsim_n$-minimal element. If we define $\succsim_n^* := \{(x, y) \in X \times X | y \succsim_n x\}$, then $\mathcal{D}(\succsim_n^*, K_n) = \mathcal{D}(\succsim_n, K_n)$ for every $n \in \mathbb{N}$. Moreover, every y_n is a $\succsim_n^*$-maximal. Thus we can apply Theorem 2.2 to conclude that y is a $\succsim^*$-maximal which is equivalent to it be a $\succsim$-minimal.

(1) $\Leftarrow$ (2). Take a sequence $\{D_n\}_{n\in\mathbb{N}}$ such that $D_n \subseteq K_n$, $D_n \in \mathcal{D}(\succsim_n, K_n)$, and $\lim_{n\to+\infty} D_n = D$. Since $\lim_{n\to+\infty} \succsim_n = \succsim$, D is a $\succsim$-domain. For each $n \in \mathbb{N}$, $\succsim_n$ is a continuous preference such that $\succsim_n \cap (K_n \times K_n)$ has connected indifference classes and K_n is $\succsim_n$-dense. Thus, by Lemma A.4, D_n is connected for every $n \in \mathbb{N}$. It follows that D is also connected by Lemma A.5 and $\succsim$-dense by Lemma A.6.

We claim that D has no exterior bounds. Suppose, seeking a contradiction, there is $x \in K$ such that $x \succsim \tilde{y}$ for every $\tilde{y} \in D$ and $x \notin D$. In fact, we must have $x \succ \tilde{y}$ because $\succsim$ is a partial order. For each D_n take $y_n \in D_n$ such that $y_n \succsim_n z$ for all $z \in D_n$. By Theorem 1 in Gorno (2018) each y_n is a maximal element in K_n. Note that $\{y_n\} \subset K_n$ and $\{y_n\}$ is compact for every $n \in \mathbb{N}$, thus by Lemma A.1 there is no loss of generality in assuming that $\{y_n\}_{n\in\mathbb{N}}$ converges to some $y \in K$. Since $y_n \in D_n$ for every n and $\{(D_n, y_n)\}_{n\in\mathbb{N}}$ converges to (D, y), we must have $y \in D$. By hypothesis we have $x \succ y$ and y is a maximal element in K, a contradiction. An analogous argument guarantees that there is no $x \in K$ such that $y \succsim x$ for every $y \in D$ and $x \notin D$. This means that D has no exterior bounds.

Furthermore, since $\succsim$ is a partial order, D contains all its indifferent alternatives. Thus, by Lemma A.4, we conclude that $D \in \mathcal{D}(\succsim, K)$. □

Theorem 2.3 provides assumptions under which condition (3) in Theorem 2.2 is necessary and sufficient for all limits of maximal or minimal elements to be maximal or minimal, respectively. An immediate application is to show that, in the case of a fixed preference, antisymmetry and midpoint continuity combined with mild additional assumptions imply condition (3).

COROLLARY 2.1. *If, in addition to the conditions of Theorem 2.1, $\succsim$ is a partial order and there exists $\underline{x} \in \bigcap_{n\in\mathbb{N}} K_n$ such that $x \succsim \underline{x}$ for every $x \in \bigcup_{n\in\mathbb{N}} K_n$, then condition (3) holds.*

This means that Theorem 2.2 does generalize Theorem 2.1 for partial orders in contexts such as consumer theory (in which consuming zero of every good is always feasible and there is no worse bundle).

A clear limitation of Theorem 2.3 is that the limit preference $\succsim$ is required to be a partial order. Example 2.5 above shows that the characterization does not hold without this assumption even if $X = [0, 1]$.

Another drawback of Theorem 2.3 is that two of its assumptions, namely $\succsim_n$-denseness of the K_n and connectedness of the relative indifference classes, refer to the specific sequence $\{(\succsim_n, K_n)\}_{n\in\mathbb{N}}$ under consideration. However, if we restrict attention to convex feasible sets and preferences over lotteries that admit an expected multi-utility representation[4], these assumptions are automatically satisfied for all sequences $\{(\succsim_n, K_n)\}_{n\in\mathbb{N}}$. This is the content of the following corollary:

COROLLARY 2.2. *Let X be the space of (Borel) probability measures on a separable metric space equipped with the topology of weak convergence. Suppose further that:*

(1) For each $n \in \mathbb{N}$, $K_n \in \mathcal{K}_X$ is convex and $\succsim_n$ is a preference that admits an expected multi-utility representation,

(2) $\succsim$ is a partial order that admits an expected multi-utility representation.

Then, the equivalence in the conclusion of Theorem 2.3 holds.

Finally, a more subtle aspect of Theorem 2.3 is that condition (2) requires that *every* convergent sequence of $\succsim$-maximal (resp. $\succsim$-minimal) elements converges to a $\succsim$-maximal (resp. $\succsim$-minimal) element. The next example illustrates this point.

EXAMPLE 2.6. Let $\succsim$ be the natural vector order on $X = [0, 1]^2$. $\succsim$ is a continuous partial order and X is $\succsim$-dense. Now, for each $n \in \mathbb{N}$, consider

$$K_n := \{(x_1, x_2) \in X | x_2 \leq n(1 - x_1)\}.$$

[4]Let C be a separable metric space of consequences and let X be the set of (Borel) probability measures (lotteries) over C equipped with the topology of weak convergence of probability measures. Similarly to Dubra, Maccheroni, and Ok (2004b), we say that a set $\mathcal{U}$ of bounded continuous functions $C \to \mathbb{R}$ constitutes an *expected multi-utility representation* for $\succsim$ whenever, for every two lotteries $x, y \in X$, $x \succsim y$ is equivalent to $\int_C u(c)dx(c) \geq \int_C u(c)dy(c)$ for all $u \in \mathcal{U}$.

Note that K_n is nonempty and compact for each $n \in \mathbb{N}$. Moreover, $\lim_{n\to+\infty} K_n = K := [0,1]^2$ and $(1,0) \in \mathcal{M}ax(\succsim, K_n)$. However, $(1,0) \notin \mathcal{M}ax(\succsim, K)$. It follows from Theorem 2.3 that there must be at least one convergent sequence $\{D_n\}_{n\in\mathbb{N}}$, where, for each $n \in \mathbb{N}$, D_n is a maximal $\succsim$-domain relative to K_n, but such that $\lim_{n\to+\infty} D_n$ is not a maximal $\succsim$-domain relative to K. In fact, taking $D_n = \{0\} \times [0,1]$ yields a specific example of such $\{D_n\}_{n\in\mathbb{N}}$.

2.6. Related literature

Considerable work has been devoted to the study of incomplete preferences.[5]. Despite the fact that continuous weak preferences are one of the two central classes of preferences in this literature, to the best of our knowledge, there is no previous work that provides positive results on the continuity properties of maximal elements for this class, as we do here.

Continuity of optimal choices is an important problem and has been studied extensively. The central result is the Maximum Theorem in Berge (1963), which is concerned with the behavior of value functions and the maximizers that attain them as parameters change continuously. Our results, in particular Theorem 2.2, extend this work by allowing for incomplete preferences. Even though its key condition is trivially satisfied when preferences are complete, Theorem 2.2 is not truly a generalization of the original Maximum Theorem because we assume that X is a metric space, whereas Berge's result is proven in a general topological space. Moreover, since assuming that preferences admit a utility representation would imply completeness, our results focus exclusively on maximal elements and make no statements about value functions.

Walker (1979) proves a generalized maximum theorem for a strict relation $\succ_\theta$ that depends on a parameter $\theta \in \Theta$ and has open graph (as a correspondence $\Theta \rightrightarrows X \times X$). Lemma 9 in Evren (2014) shows that if X is a space of lotteries and $\succ$ is open, then $\mathcal{M}ax(\succsim, K)$ is relatively closed in K and the correspondence $K \rightrightarrows \mathcal{M}ax(\succsim, K)$ is upper hemicontinuous, even if K is neither convex nor compact. However, neither of these two results bears significance for the class of incomplete continuous weak preferences considered in this chapter. The reason is that, as long as X is connected, every incomplete continuous weak preference that has an open strict part must be

[5]The list is long. A few examples in chonological order are Aumann (1962),Peleg (1970),Ok (2002), Dubra, Maccheroni, and Ok (2004b), Eliaz and Ok (2006), Dubra (2011), Evren and Ok (2011), Ok, Ortoleva, and Riella (2012), Evren (2014), Riella (2015), Gorno (2017), and Gorno (2018).

trivial (see Schmeidler (1971)) and, as a result, satisfy $\mathcal{M}ax\,(\succsim, K) = K$ for every $K \subseteq X$.

2.7. Discussion

The present chapter provides three major results that expand the scope of Berge's Maximum Theorem to allow for incomplete preferences. Theorem 2.1 is based on a simple continuity condition, but its applicability is somewhat limited since it requires convex feasible sets and a fixed preference.

Theorem 2.2 does not have the aforementioned limitations. The result depends crucially on its condition (3), a form of upper hemicontinuity of the mapping between preferences-feasible sets pairs and the corresponding collection of maximal domains of comparability. Since Gorno (2018) shows that every maximal element is the best element in some maximal domain and vice-versa, convergence of maximal domains permits the application of a Berge-type of argument to ensure the convergence of maximal elements through the convergence of local best elements, where the term "local" here means "relative to a maximal domain".

Finally, Theorem 2.3 describes a more specific setting in which condition (3) in Theorem 2.2 is necessary and sufficient for minimality and maximality to be preserved when taking limits.

We believe that these results constitute a step forward towards understanding convergence of maximal elements without completeness and open at least three avenues for future research. First, the abstract nature Theorem 2.2 suggests to look for additional sets of assumptions which are sufficient for its condition (3) to hold. Second, the equivalence in Theorem 2.3 might be true under weaker assumptions. Third, we currently do not know whether our results remain true in a general topological space (which is the environment in which Berge's Maximum Theorem is formulated).

CHAPTER 3

Connected Incomplete Preferences

3.1. Introduction

The standard model of choice in economics is the maximization of a complete and transitive preference relation over a fixed set of alternatives. While completeness of preferences is usually regarded as a strong assumption, weakening it requires care to ensure that the resulting model still has enough structure to yield interesting results. This chapter takes a step in this direction by studying the class of "connected preferences", that is, preferences that may fail to be complete but have connected maximal domains of comparability.[1]

We offer four new results. Theorem 3.1 identifies a basic necessary condition for a continuous preference to be connected in the sense above, while Theorem 3.2 provides sufficient conditions. Building on the latter, Theorem 3.3 characterizes the maximal domains of comparability. Finally, Theorem 3.4 presents conditions that ensure that maximal domains are arc-connected.

Methodologically, our contribution provides an incomplete preference perspective on a theoretical literature relating basic assumptions on preferences and the space of alternatives over which these preferences are defined. For example, Schmeidler (1971) shows that every nontrivial preference on a connected topological space which satisfies seemingly innocuous continuity conditions must be complete. In a recent article, Khan and Uyank (2019) revisit Schmeidler's theorem and link it to the results in Eilenberg (1941), Sonnenschein (1965), and Sen (1969), providing a thorough analysis of the logical relations between the form of continuity assumed by Schmeidler, completeness, transitivity, and the connectedness of the space.

In particular, Theorem 4 in Khan and Uyank (2019) implies a converse to Schmeidler's theorem: if every strongly nontrivial Schmeidler preference is complete, the underlying space must be connected. We provide a different kind of converse: every compact space that admits at least one complete and gapless Schmeidler preference with connected indifference classes must be connected.

[1] Gorno (2018) examines the maximal domains of comparability of a general preorder.

3.2. Preliminaries

Let X be a (nonempty) set of alternatives equipped with some topology. A *preference* is a reflexive and transitive binary relation on X. For the rest of this chapter, we consider a fixed preference $\succsim$.

$\succsim$ is *complete* on a set $A \subseteq X$ if $A \times A \subseteq \succsim \cup \precsim$. The set A is a *domain* if $\succsim$ is complete on A. If A is a domain such that there exists no larger domain containing it, then A is a *maximal domain.*

$\succsim$ is *Debreu continuous* if $\{y \in X | y \succsim x\}$ and $\{y \in X | x \succsim y\}$ are closed sets for every $x \in X$. $\succsim$ has *connected indifference classes* if $\{y \in X | y \sim x\}$ is connected for every $x \in X$.

The set $A \subseteq X$ *contains every indifferent alternative* if $x \in A$, $y \in X$, and $x \sim y$ implies $y \in A$. *A has no exterior bound* if $x \succsim A \succsim y$ implies $x, y \in A$.

3.3. Connected preferences

The main concept of this chapter is embedded in the following definition:

Definition 3.1. $\succsim$ is *connected* if every maximal domain is connected.

We will restrict attention to preferences that are not only connected, but also Debreu continuous. As a result, maximal domains will be necessarily closed (see Theorem 1 in Gorno (2018)).

3.3.1. A necessary condition. A natural first step towards a characterization of connected preferences is to obtain a simple necessary condition.

Definition 3.2. $\succsim$ is *gapless* if, for every $x, y \in X$, $x \succ y$ implies that there exists $z \in X$ such that $x \succ z \succ y$.

The notion of gapless preferences is not really new; its content coincides with a specific definition of order-denseness for sets.[2] We now prove our first result:

Theorem 3.1. *If $\succsim$ is Debreu continuous and connected, then $\succsim$ is gapless.*

Proof. Suppose, seeking a contradiction, that $\succsim$ is not gapless. Then, there exist alternatives $x, y \in X$ such that $x \succ y$ and no $z \in X$ satisfies $x \succ z \succ y$. By Lemma 1 in Gorno (2018), there exists a maximal domain D such that $\{x, y\} \subseteq D$.

[2] X is said to be $\succsim$-*dense* if for every $x, y \in X$ satisfying $x \succ y$ there exists $z \in X$ such that $x \succ z \succ y$ (see Ok (2007), p. 92). Evidently, X is $\succsim$-dense if and only if $\succsim$ is gapless. We should perhaps note that there are multiple distinct de nitions of order-denseness in the literature and that the terminology has not necessarily been consistent.

Define $A := \{z \in D | z \succsim x\}$ and $B := \{z \in D | y \succsim z\}$. Clearly, A and B are nonempty, $A \cap B = \emptyset$, and $A \cup B = D$. Moreover, since $\succsim$ is Debreu continuous, A and B are closed relative to D. It follows that D is not connected, a contradiction. □

It is easy to see that not every Debreu continuous and gapless preference is connected:

EXAMPLE 3.1. Let $X = [-1, 1]$ and $\succsim = \{(x, y) \in X^2 | x = y \vee x^2 = y^2 = 1\}$. Then, the preference $\succsim$ is Debreu continuous and gapless, but not connected (the maximal domain $\{-1, 1\}$ is not a connected set).

3.3.2. A sufficiency theorem. We already know that every Debreu continuous and connected preference must be gapless. In this section, we provide a set of assumptions which constitute a sufficient condition for a preference to be connected.

THEOREM 3.2. *If X is compact and $\succsim$ is a Debreu continuous and gapless preference with connected indifference classes, then $\succsim$ is connected.*

PROOF. Suppose, seeking a contradiction, that there is a maximal domain D that is not connected. Then, there exist disjoint nonempty sets A and B such that $A \cup B = D$ and both are closed relative to D. Since $\succsim$ is Debreu continuous Proposition 1 in Gorno (2018) implies that D is closed in X, hence A and B are also closed in X. Moreover, since X is compact, A and B are compact as well. Let $\overline{x}_A$ and $\overline{x}_B$ be the best elements in A and B, respectively. Since D is a domain, $\overline{x}_A$ and $\overline{x}_B$ are comparable, which means that either $\overline{x}_A \sim \overline{x}_B$, $\overline{x}_A \succ \overline{x}_B$, or $\overline{x}_B \succ \overline{x}_A$. Suppose first that $\overline{x}_A \sim \overline{x}_B$ and consider the indifference class $I := \{x \in X | x \sim \overline{x}_A\}$. Note that $I \subseteq D$, because D is a maximal domain. Hence, the sets $I_1 := A \cap I$ and $I_2 := B \cap I$ are nonempty, disjoint, and closed relative to I, which contradicts the assumption that $\succsim$ has connected indifference classes. Suppose now that $\overline{x}_A \succ \overline{x}_B$ (the remaining case is symmetric). Define the set $C := \{x \in A | x \succsim \overline{x}_B\}$. C is nonempty (as $\overline{x}_A \in C$) and compact. Let $\underline{x}_C$ be the worst element in C. It is easy to check that $\underline{x}_C \succ \overline{x}_B$. Since $\succsim$ is gapless, there exists $z \in X$ such that $\underline{x}_C \succ z \succ \overline{x}_B$. It is easy to verify that $z \notin A$ and $z \notin B$. Hence, $z \notin D$. Moreover, $D \cup \{z\}$ is a domain, contradicting the assumption that D is a maximal domain. □

The following example identifies an important class of connected preferences:

EXAMPLE 3.2. Let X be the set of Borel probability measures (lotteries) on a compact metric space of prizes Z, equipped with the topology of weak convergence.

Following Dubra, Maccheroni, and Ok (2004b), we say that the preference $\succsim$ is an *expected multi-utility preference* if there exists a set $\mathcal{U}$ of continuous functions $Z \to \mathbb{R}$ such that $x \succsim y$ if and only if

$$\int_Z u dx \geq \int_Z u dy$$

holds for all $u \in \mathcal{U}$. It is easy to verify that all the assumptions of Theorem 3.2 hold. Thus, $\succsim$ is connected.

3.4. Characterization of maximal domains

Building on Theorem 3.2, we can offer a useful characterization of the maximal domains:

THEOREM 3.3. *Assume X is compact and $\succsim$ is Debreu continuous, gapless, and has connected indifference classes. Then, a set $A \subseteq X$ is a maximal domain if and only if it is a connected domain that contains every indifferent alternative and has no exterior bound.*

PROOF. We start establishing sufficiency through the following lemma:

LEMMA 3.1. *Every connected domain that contains every indifferent alternative and has no exterior bound is a maximal domain.*

PROOF. Suppose, seeking a contradiction, that D is a domain that contains every indifferent alternative, has no exterior bound, but it is not a maximal domain, then by Lemma 1 in Gorno (2018) exists D', a maximal domain, such that $D \subset D'$. Take $x \in D' \backslash D$. Since D has no exterior bounds there are $y, z \in D$ such that $y \succ x \succ z$. Define $D_1 := \{w \in D | w \succsim x\}$ and $D_2 := \{w \in D | x \succsim w\}$. D_1 and D_2 are nonempty since $y \in D_1$ and $z \in D_2$. Also, $D_1 \cup D_2 = D$ because $x \in D'$ and D' is a domain that contains D. Moreover, $D_1 \cap D_2 = \emptyset$. If this intersection was not empty, there would be $w \in D$ such that $x \sim w$, which would contradict that D contains every indifferent alternative. Finally, D_1 and D_2 are closed relative to D because $\succsim$ is Debreu continuous. It follows that $\{D_1, D_2\}$ is a nontrivial partition of D by closed sets. We conclude that D is not connected, which is a contradiction. □

Now we turn to necessity. It is easy to show that every maximal domain contains every indifferent alternative and has no exterior bound. Moreover, since $\succsim$ satisfies the assumptions of Theorem 3.2, every maximal domain is connected. □

We finish this section, discussing the two additional assumptions employed in Theorem 3.3.

3.4.1. X **is compact.** Compactness of X cannot be dispensed with, as the following example shows.

EXAMPLE 3.3. Let $X = \{-1\} \cup [0, 1)$ and $\succsim = \{(x, y) \in X^2 | x = -1 \vee x \geq y \geq 0\}$. Then, X is bounded, locally compact and σ-compact, but fails to be compact. Moreover, $\succsim$ is complete, Debreu continuous, and gapless. However, the only maximal domain is X itself that is not connected.

3.4.2. Connected indifferent classes. On the one hand, the assumption that indifferent classes are connected is not strictly necessary for the conclusion of Theorem 3.3. That is, there are examples failing this condition in which the equivalence in the theorem holds:

EXAMPLE 3.4. Let $X = [-1, 1]$ and $\succsim = \{(x, y) \in X^2 | x^2 \geq y^2\}$.

On the other hand, it is a tight condition: there are examples that violate it, satisfy the remaining conditions, and for which the equivalence in the theorem fails to hold:

EXAMPLE 3.5. Let $X = \{-1\} \cup [0, 1]$ and $\succsim = \{(x, y) \in X^2 | x^2 \geq y^2\}$.

There is a well-known axiom introduced by Dekel (1986) that ensures that indifference classes are connected. Assuming that X is convex, we say that $\succsim$ satisfies *betweenness* if $x \succsim y$ implies $x \succsim \alpha x + (1 - \alpha)y \succsim y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. Prominent examples of preferences satisfying betweenness include preferences satisfying the independence axiom (such as expected utility or the expected multi-utility preferences studied in Dubra, Maccheroni, and Ok (2004b)) and also preferences exhibiting disappointment aversion as in Gul (1991). The following lemma shows that betweenness implies connected indifference classes.

LEMMA 3.2. *If X is convex and $\succsim$ satisfies betweenness, then $\succsim$ has connected indifference classes.*

PROOF. Take any $x, y \in X$ such that $x \sim y$ and $\alpha \in [0, 1]$. Define $z := \alpha x + (1 - \alpha)y$. Since $x \succsim y$ and $y \succsim x$, by betweenness, we have $x \succsim z \succsim y$ and $y \succsim z \succsim x$ and, so $z \sim y$. It follows that each indifference class is convex, thus connected. □

We should note that, if X is convex and $\succsim$ is a Debreu continuous preference that satisfies betweenness, then $\succsim$ does not only possess connected indifferent classes, but is also necessarily gapless. This fact makes the application of Theorem 3.2 and Theorem 3.3 to preferences satisfying betweenness quite direct.

3.5. Arc-connected preferences

In some cases, it can be useful to strengthen the notion of connectedness to arc-connectedness:

DEFINITION 3.3. $\succsim$ is *arc-connected* if every maximal domain is arc-connected.

Every arc-connected preference is connected, but the converse does not generally hold. To see this it suffices to take X to be any space that is connected but not arc-connected[3] and consider $\succsim = X \times X$, that is, universal indifference.

In the particular case of antisymmetric preferences (*i.e.*, partial orders) on a metrizable space, we can strengthen the conclusion of Theorem 3.2:

THEOREM 3.4. *If X is a compact metrizable space and $\succsim$ is a Debreu continuous, gapless, and antisymmetric preference, then $\succsim$ is arc-connected.*

PROOF. Let D be a maximal domain. Since $\succsim$ is Debreu continuous and X is compact and metrizable, Theorem 1 in Gorno (2018) implies that D is compact and metrizable, hence second countable. Because $\succsim$ is complete and Debreu continuous on D, there exists a continuous utility representation $u : D \to \mathbb{R}$.

Since $\succsim$ is antisymmetric, its indifference classes are singletons, hence connected. By Theorem 3.2, D is connected. It follows that $u(D)$ is connected and compact, thus a compact interval. Without loss of generality, we can assume that $u(D) = [0, 1]$. Since $\succsim$ is antisymmetric, u is a continuous bijection. Since X is compact and $[0, 1]$ is Hausdorff, u is actually an homeomorphism between D and $[0, 1]$. It follows that D is arc-connected. as desired. $\square$

3.6. Applications

3.6.1. First-order stochastic dominance. Suppose X is the set of cumulative distribution functions (CDFs) over a compact interval $[0, \overline{z}]$ (endowed with the topology of weak convergence of the associated probability measures). Let $\geq_1$ denote the first-order stochastic dominance relation on X, that is, $F \geq_1 G$ if and only if $F(z) \leq G(z)$ for all $z \in [0, \overline{z}]$.

[3]A well-known example is the closed topologist's sine curve, which is also compact.

PROPOSITION 3.1. *$\geq_1$ is arc-connected. Moreover, a subset of X is a maximal domain of $\geq_1$ if and only if it is the image of a $\geq_1$-increasing arc joining the degenerate CDFs associated with* 0 *and* $\overline{z}$.

PROOF. X is a compact metrizable space (it is metrized by the Lévy metric) and $\geq_1$ is Debreu continuous, gapless, and antisymmetric. Thus, by Theorem 3.4, $\geq_1$ is arc-connected. Since every arc-connected set is connected, Theorem 3.3 implies the desired equivalence. □

An analogous result holds for second-order stochastic dominance.

3.6.2. Schmeidler preferences. Schmeidler (1971) shows that, in a connected space, every nontrivial preference satisfying seemingly innocuous continuity conditions must be complete. In this section, we explore the implications of his assumptions in spaces that are not connected.

We start by formulating the class of preferences which are the subject of Schmeidler's theorem:

DEFINITION 3.4. A preference $\succsim$ is a *Schmeidler preference* if it is Debreu continuous and the sets $\{y \in X | x \succ y\}$ and $\{y \in X | y \succ x\}$ are open for all $x \in X$.

The following definition captures a property that generalizes the conclusion of Schmeidler's theorem in terms of maximal domains:

DEFINITION 3.5. A preference is *decomposable* if every maximal domain is either a connected component or an indifference class.

Note that, when X is connected, every nontrivial decomposable preference is complete. More generally, any two distinct maximal domains of a decomposable preference must necessarily be disjoint. As a result, if a decomposable preference is locally nonsatiated, then no maximal domain can be trivial or, equivalently, every maximal domain must be a connected component.

We can now state the main result of this section:

PROPOSITION 3.2. *Let X be compact and let $\succsim$ be a Schmeidler preference with connected indifference classes. Then, $\succsim$ is decomposable if and only if $\succsim$ is gapless.*

PROOF. To prove necessity, assume that $\succsim$ is gapless. Since $\succsim$ is a Schmeidler preference, Proposition 10 in Gorno (2018) implies that every nontrivial connected component is contained in a maximal domain. Moreover, because $\succsim$ is a gapless preference on a compact space, every maximal domain is connected by Theorem 3.3. It follows that every nontrivial maximal domain is a connected component. Finally, since trivial maximal domains must be indifference classes, $\succsim$ is decomposable.

For sufficiency, note that, since $\succsim$ is decomposable and has connected indifference classes, every maximal domain is connected. Thus, Theorem 3.1 implies that $\succsim$ is gapless. □

Note that every Debreu continuous and complete preference is a Schmeidler preference. In that particular case, we have the following

COROLLARY 3.1. *Let X be compact and let $\succsim$ be a Debreu continuous and complete preference with connected indifference classes. Then, $\succsim$ is gapless if and only if X is connected.*

Schmeidler (1971) shows that if X is connected, then every nontrivial Schmeidler preference must be complete. Khan and Uyank (2019) prove the converse and obtain the following characterization: X is connected if and only if every nontrivial Schmeidler preference is complete. The corollary above implies a different characterization for compact spaces: provided X is compact, X is connected if and only if there exists at least one complete, gapless, and Debreu continuous preference with connected indifference classes.[4]

[4]Note that, if X is connected, the trivial preference that declares all alternatives indi erent satis es all the desired properties.

APPENDIX A

Technical lemmas

The proof of Theorem 2.2 requires some results about Hausdorff convergence. In the following two lemmas (M, d) is any metric space and $\mathcal{K}_M$ (resp. $\mathcal{F}_M$) is the collection of all nonempty compact (resp. closed) subsets of M.

LEMMA A.1. *Let $\{K_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathcal{K}_M$ with limit $K \in \mathcal{K}_M$. Then, every sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}_M$ such that $A_n \subseteq K_n$ for all $n \in \mathbb{N}$ has a subsequence which converges to a nonempty compact subset of K.*

PROOF. Let $d^H : \mathcal{K}_M \times \mathcal{K}_M \to \mathbb{R}_+$ denote the Hausdorff distance and let $\mathcal{K}_K$ be the collection of all nonempty compact subsets of K. Note that $\left(\mathcal{K}_M, d^H\right)$ is a metric space, $\mathcal{K}_K$ is compact in the (relative) Hausdorff metric topology, and $d^H(A_n, \cdot)$ is continuous on $\mathcal{K}_K$. For each $n \in \mathbb{N}$, let $B_n \in \arg\min_{\tilde{B}\in\mathcal{K}_K} d^H(A_n, \tilde{B})$. I now claim that, for each $n \in \mathbb{N}$, we have

$$d^H(A_n, B_n) \leq d^H(K_n, K).$$

To prove this claim, note that, since $\{y\} \in \mathcal{K}_K$ for all $y \in K$, we have

$$d^H(A_n, B_n) \leq d^H(A_n, \{y\}) = \max_{x\in A_n} d(x, y)$$

for all $y \in K$. Defining $y^*(x) \in \arg\min_{y\in K} d(x, y)$ for each $x \in A_n$, we have

$$\begin{aligned} d^H(A_n, B_n) &\leq \max_{x\in A_n} d(x, y^*(x)) = \max_{x\in A_n} \min_{y\in K} d(x, y), \\ &\leq \max_{x\in K_n} \min_{y\in K} d(x, y) \leq d^H(K_n, K) \end{aligned}$$

as desired. It follows that $\lim_{n\to+\infty} d^H(A_n, B_n) \leq \lim_{n\to+\infty} d^H(K_n, K_n) = 0$.

Since $\mathcal{K}_K$ is compact, the sequence $\{B_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, say $\{B_{n_h}\}_{h\in\mathbb{N}}$. Let $A := \lim_{h\to+\infty} B_{n_h} \in \mathcal{K}_K$. Since $\lim_{n\to+\infty} d^H(A_n, B_n) = 0$ and $\lim_{h\to+\infty} B_{n_h} = A$, the triangle inequality $d^H(A_{n_h}, A) \leq d^H(A_{n_h}, B_{n_h}) + d^H(B_{n_h}, A)$ implies $\lim_{h\to+\infty} d^H(A_{n_h}, A) = 0$. This means that $\{A_{n_h}\}_{h\in\mathbb{N}}$ converges to A, completing the proof. □

Lemma A.2. *Denote by $\mathcal{F}_M$ the collection of all nonempty closed subsets of M and let $\{(F_n, x_n)\}_{n\in\mathbb{N}}$ be a convergent sequence on $\mathcal{F}_M \times M$ with limit $(F, x) \in \mathcal{F}_M \times M$ and such that $x_n \in F_n$ for every $n \in \mathbb{N}$. Then, $x \in F$.*

Proof. Suppose, seeking a contradiction, that $x \notin F$. Since F is closed, there exists $\epsilon > 0$ such that $\{y \in M | d(x, y) \leq \epsilon\} \cap F = \emptyset$. Hence, $\inf_{y\in F} d(x, y) > \epsilon/2$. By the triangule inequality, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

for all $y \in F$ and all $n \in \mathbb{N}$. Therefore

$$\inf_{y\in F} d(x, y) \leq \inf_{y\in F} \{d(x, x_n) + d(x_n, y)\} = d(x, x_n) + \inf_{y\in F} d(x_n, y) \leq d(x, x_n) + d^H(F_n, F)$$

for all $n \in \mathbb{N}$, where we used

$$\inf_{y\in F} d\,(x_n, y) \leq \sup_{x\in F_n} \inf_{y\in F} d(x, y) \leq d^H(F_n, F)$$

Taking limits we conclude that $\inf_{y\in F} d(x, y) = 0$, a contradiction. □

Lemma A.3. *Consider a convergent sequence $\{(\succsim_n, K_n, x_n, y_n)\}_{n\in\mathbb{N}}$ in $\mathcal{P} \times \mathcal{K}_X \times X \times X$ with limit $(\succsim, K, x, y) \in \mathcal{P} \times \mathcal{K}_X \times X \times X$. Then:*

(1) $x_n \in K_n$ for all $n \in \mathbb{N}$ implies $x \in K$.
(2) $x_n \succsim_n y_n$ for all $n \in \mathbb{N}$ implies $x \succsim y$.

Proof. The first part follows from Lemma A.1 by taking $M = X$ and $A_n = \{x_n\}$ for each $n \in \mathbb{N}$, since every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in K$. The second part follows from Lemma A.2 by taking $M = X \times X$ and noting that $x_n \succsim_n y_n$ means $(x_n, y_n) \in \succsim_n$. □

The next three lemmas are used in the proof of Theorem 2.3. In what follows, K is an element of $\mathcal{K}_X$ and $\succsim$ is a continuous preference on X. A set $A \subseteq K$ *has no exterior bound in* K if, for every $x, y \in K$, $x \succsim A \succsim y$ implies $x, y \in A$.

Lemma A.4. *Assume that K is $\succsim$-dense and the indifference classes of $\succsim$ in K are connected. Then, a subset of K is a maximal $\succsim$-domain relative to K if and only if it is a connected $\succsim$-domain relative to K which has no exterior bound in K.*

Proof. Since K is compact and $\succsim \cap (K \times K)$ is a continuous preference on K with connected indifference classes, the result follows from Theorem 4 in Gorno and Rivello (2020). □

LEMMA A.5. *Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_X$ such that K_n is connected for every n and $\lim_{n\to+\infty} K_n = K$, then K is connected.*

PROOF. Suppose, seeking a contradiction, that K is not connected. Then, there exist disjoint nonempty sets A and B which are closed in K and satisfy $A \cup B = K$. For any $\epsilon > 0$, define $A^\epsilon := \{x \in X | d(x, A) < \epsilon\}$ and $\bar{A}^\epsilon$ as its closure. Define B^ϵ and $\bar{B}^\epsilon$ analogously. Define $K^\epsilon := A^\epsilon \cup B^\epsilon$ and $\bar{K}^\epsilon$ as its clousure.

Since K is closed, A and B are also closed in X, which is a normal space. Then, there exists $\bar{\epsilon} > 0$ such that, for every $\epsilon \in (0, \bar{\epsilon}]$, we have $A^\epsilon \cap B^\epsilon = \emptyset$. Fix $\epsilon = \bar{\epsilon}/2$, then $\bar{A}^\epsilon \cap \bar{B}^\epsilon = \emptyset$. Because $\lim_{n\to+\infty} K_n = K$ there is $N_\epsilon \in \mathbb{N}$ such that $n \geq N_\epsilon$ implies $K_n \subseteq \bar{K}^\epsilon$. Now define $A_n := K_n \cap \bar{A}^\epsilon$ and $B_n := K_n \cap \bar{B}^\epsilon$. It is easy to see that $A_n \cap B_n = \emptyset$, $A_n \cup B_n = K_n$, and $A_n, B_n \in \mathcal{K}_X$. It follows that K_n is not connected, a contradiction. □

LEMMA A.6. *Let $\{(\succsim_n, K_n)\}_{n\in\mathbb{N}}$ be a converging sequence in $\mathcal{P} \times \mathcal{K}_X$ with limit $(\succsim, K) \in \mathcal{G} \times \mathcal{K}_X$. If, for every $n \in \mathbb{N}$, K_n is $\succsim_n$-dense and all indifference classes of $\succsim_n$ in K_n are connected, then K is $\succsim$-dense.*

PROOF. Suppose, seeking a contradiction, that K is not $\succsim$-dense. Then, there exist $x, y \in K$ such that $x \succ y$ and there is no $z \in K$ that satisfies $x \succ z \succ y$. Take $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $x_n, y_n \in K_n$ for every $n \in \mathbb{N}$, $\lim_{n\to+\infty} x_n = x$, and $\lim_{n\to+\infty} y_n = y$. Define $M_n := \{z \in K_n | x_n \succsim_n z \succsim_n y_n\}$. Note that $M_n \in \mathcal{K}_X$ and $M_n \subseteq K_n$ for every $n \in \mathbb{N}$, so Lemma A.1 implies that $\{M_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Thus, we can assume without loss of generality that $\{M_n\}_{n\in\mathbb{N}}$ itself converges and define $M := \lim_{n\to+\infty} M_n$.

We claim that M_n is connected for each $n \in \mathbb{N}$. Suppose, seeking a contradiction, that M_n is not connected for some $n \in \mathbb{N}$. Then there should exist disjoint nonempty sets A and B which are closed in M_n and satisfy $A \cup B = M_n$. Without loss, assume that $x_n \in A$. Since B is compact and $\succsim_n$ is continuous, there exists at least one $\succsim_n$-maximal element in B, call it $\overline{x}_B$. Define $C := \{z \in A | z \succsim_n \overline{x}_B\}$. Note that C is also compact and nonempty ($x_n \in C$), so we can take $\underline{x}_C$, one of its $\succsim_n$-minimal elements. We will now show that $\underline{x}_C \succ_n \overline{x}_B$. Define $I := \{z \in K_n | z \sim_n \bar{x}_B\}$. Since $I \subseteq M_n$, both $I \cap B$ and $I \cap A$ are closed sets which satisfy $(I \cap A) \cup (I \cap B) = I$ and $(I \cap A) \cap (I \cap B) = \emptyset$. Since I is assumed to be connected and $\bar{x}_B \in I \cap B$ it must be that $I \cap A = \emptyset$, proving that $\underline{x}_C \succ_n \overline{x}_B$. Define $D := \{z \in M_n | \underline{x}_C \succ_n z \succ_n \overline{x}_B\}$. If there is $z \in D$, then $z \succ_n \overline{x}_B$ implies $z \notin B$. Moreover $\underline{x}_C \succ_n z \succ_n \overline{x}_B$ implies that

$z \notin C$, so $z \notin A$ either. It follows that D must be empty, which is a contradiction with K_n being $\succsim_n$-dense. We conclude that M_n is connected.

On the one hand, since $\{M_n\}_{n\in\mathbb{N}}$ is a sequence of connected sets in $\mathcal{K}_X$, M is connected by Lemma A.5. On the other hand, we claim that $M = \{x, y\}$. It is easy to see that $\{x, y\} \subseteq M$. To prove the other inclusion take any sequence $\{z_n\}_{n\in\mathbb{N}}$ converging to $z \in M$ and such that $z_n \in M_n$ for every $n \in \mathbb{N}$. Then $x_n \succsim_n z_n \succsim_n y_n$ implies $x \succsim z \succsim y$. But, since we initially assumed that there is no $z \in K$ that satisfies $x \succ z \succ y$, we must necessarily have that either $x \sim z$ or $z \sim y$. Furthermore, $\succsim$ antisymmetric implies that $x = z$ or $z = y$. Hence, M must be equal to $\{x, y\}$.

We conclude that M must simultaneously be connected and equal to $\{x, y\}$, which yields the desired contradiction. $\square$

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